Some results on self-orthogonal and self-dual codes

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Abstract

We use generator matrices $G$ satisfying $GG^T = aI + bJ$ over $\mathbb{Z}_k$ to obtain linear self-orthogonal and self-dual codes. We give a new family of linear self-orthogonal codes over $GF(3)$ and $\mathbb{Z}_4$ and a new family of linear self-dual codes over $GF(3)$.

Key words and phrases: Self-orthogonal, self-dual, codes, construction, conference matrix, projective plane.

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1 Introduction

A linear code $C$ of length $n$ over $\mathbb{Z}_k$ (or a $\mathbb{Z}_k$-code of length $n$) is a $\mathbb{Z}_k$-submodule of $\mathbb{Z}_k^n$. If $k = p$ is prime then $\mathbb{Z}_p = GF(p)$ and a linear code of length $n$ is a subspace of $GF(p)$. An element of $C$ is called a codeword. We define the inner product on $\mathbb{Z}_k^n$ by $x \cdot y = x_1y_1 + \cdots + x_ny_n$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. The dual code $C^\perp$ of $C$ is defined as $C^\perp = \{ v \in \mathbb{Z}_k^n | v \cdot w = 0 \ \text{for all} \ w \in C \}$.
\[w \in C\}. \text{ A code } C \text{ is } \textit{self-dual} \text{ if } C = C^\perp. \text{ The Hamming weight } (\text{wt}(c)) \text{ of a codeword } c \text{ is the number of non-zero components in the codeword. The } \text{minimum weight} \text{ of a code } C \text{ is the smallest weight among all codewords of } C. \text{ The minimum distance of a linear code } C \text{ is its minimum weight. We say that self-dual codes with the largest minimum weight among self-dual codes of that length are } \textit{optimal}. \text{ A linear code over } GF(p) \text{ of length } n \text{ with } k \text{ independent rows in its generator matrix will be denoted as } [n, k; p]. \text{ Furthermore, if its minimum distance is } d \text{ it will be denoted as } [n, k, d; p].

Two codes over \(Z_k\) are said to be \textit{equivalent} if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates.

There has been a large amount of research recently devoted to self-orthogonal and self-dual codes over the ring \(Z_4\), \([1, 3, 5, 7]\). Patrick Solé’s remark that the orthogonality of Hadamard matrices can naturally be interpreted as \(Z_4\)-orthogonality was investigated in \([4]\). These self-orthogonal and self-dual codes over \(Z_4\) were obtained from equivalence classes of Hadamard matrices.

2 The constructions

We give a general theorem which will be used later in the paper.

**Theorem 1** Suppose \(A\) and \(B\) are two matrices of order \(n\) over \(Z_k\) satisfying
\[AA^T + BB^T = sI + rJ\]
where \(s \equiv r \equiv 0(\text{mod } k)\). Then
\[G = [A \ B]\]
genarates a linear self-orthogonal code over \(Z_k\), of length \(2n\) and with \(m, m \leq \frac{n}{2}\) independent rows in its generator matrix. \(\Box\)

The next corollary is a generalization of a construction given by Georgiou and Koukouvinos \([6]\).

**Corollary 1** Suppose \(A\) and \(B\) are two matrices of order \(n\) over \(Z_k\) satisfying
\[AA^T = a_1I + a_2J \text{ and } BB^T = b_1I + b_2J\]
where \( a_1 + b_1 \equiv a_2 + b_2 \equiv 0 \pmod{k} \). Then
\[
G = [A \ B]
\]
generates a linear self-orthogonal code of length \( 2n \) and with \( m \) independent rows in its generator matrix, over \( \mathbb{Z}_k \), \( m \leq \frac{n}{2} \).

\[\square\]

**Theorem 2** Suppose \( A \) and \( B \) are two matrices of order \( n \) over \( \mathbb{Z}_k \) satisfying
\[
AA^T = a_1I + a_2J \text{ and } BB^T = b_1I + b_2J
\]
where \( a_2 + b_2 \equiv 0 \pmod{k} \) and \( a_1 + b_1 + a \equiv 0 \pmod{k} \) for some \( a \in \mathbb{Z}_k \). Then
\[
G_2 = \begin{bmatrix}
aI_{2n} & A & B \\
& B^T & -A^T
\end{bmatrix}
\]
generates a linear self-dual code of length \( 4n \) and with \( 2n \) independent rows in its generator matrix, over \( \mathbb{Z}_k \).

\[\square\]

**Example 1** (i) Set \( A = B = \text{circ}(1, 1, 1, 1, 0) \). We have that
\[
AA^T = BB^T = I + 3J.
\]
Then
\[
G_2 = \begin{bmatrix}
I_{2n} & A & B \\
& B^T & -A^T
\end{bmatrix}
\]
generates an \([20, 10, 6; 3]\) extremal self-dual code with weight enumerator
\[
W(z) = 1 + 120z^6 + 4260z^9 + 26280z^{12} + 25728z^{15} + 2560z^{18}.
\]

(ii) Set \( A = \text{circ}(-2, -2, 0, -1, 0) \) and \( B = \text{circ}(-1, -1, -1, -1, 1) \). We have that \( AA^T = 5I + 4J \) and \( BB^T = 4I + J \). Then
\[
G_2 = \begin{bmatrix}
I & A & B \\
& B^T & -A^T
\end{bmatrix}
\]

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generates an $[20, 10, 8; 5]$ extremal self-dual code with weight enumerator
\[ W(z) = 1 + 1280z^8 + 3200z^9 + 24848z^{10} + 58560z^{11} + 248480z^{12} +
+ 164960z^{13} + 1175840z^{14} + 1568000z^{15} + 226720z^{16} +
+ 1896720z^{17} + 1398960z^{18} + 541760z^{19} + 115776z^{20}. \]

(ii) Set $A = \text{circ}(-2, -2, 0, -1, 0)$ and $B = \text{circ}(-1, -1, -1, -1, 1)$. We have that $AA^T = 5I + 4J$ and $BB^T = 4I + J$. Then
\[ G = [A \ B] \]
generates an $[10, 5, 4; 5]$ self-dual code with weight enumerator
\[ W(z) = 1 + 40z^4 + 44z^5 + 220z^6 + 760z^7 + 940z^8 + 740z^9 + 380z^{10}. \]

For the SBIBDs we use in the remainder of this paper, we refer the reader to the book of Beth, Jungnickel and Lenz [2]. By $A = \text{SBIBD}(v, k, \lambda)$ we denote the $v \times v$ $(0, 1)$ incidence matrix of the $\text{SBIBD}(v, k, \lambda)$.

**Example 2**

1. There exist $A = \text{SBIBD}(31, 10, 3)$ and $B = \text{SBIBD}(31, 15, 7)$, so $[A \ B]$ generates a linear self-orthogonal code of length 62 and with $k_1$ independent rows in its generator matrix, over $GF(5)$ with minimum distance $d_1$ as
\[ AA^T = 7I + 3J \text{ and } BB^T = 8I + 7J. \]

2. There exist $A = \text{SBIBD}(71, 15, 3)$ and $B = \text{SBIBD}(71, 21, 6)$, so $[A \ B]$ generates a linear self-orthogonal code of length 142 and with $k_2$ independent rows in its generator matrix, over $GF(3)$ with minimum distance $d_2$ as
\[ AA^T = 12I + 3J \text{ and } BB^T = 15I + 6J. \]

3. There exist $A = \text{SBIBD}(133, 33, 8)$ and $B = \text{SBIBD}(133, 12, 1)$, so $[A \ B]$ generates a linear self-orthogonal code of length 266 and with $k_3$ independent rows in its generator matrix, over $GF(3)$ with minimum distance $d_3$ as
\[ AA^T = 25I + 8J \text{ and } BB^T = 11I + J. \]

\[ \square \]
In the next theorems we use specific families to find linear self-orthogonal codes. We combine skew-Hadamard matrices or conference matrices with incidence matrices of projective planes to construct some linear self-orthogonal codes over $\mathbb{Z}_k$.

Details on skew-Hadamard matrices and conference matrices required for the next theorem can be found in Seberry and Yamada [9]. Appropriate details of the incidence matrices of projective planes can be found in Ryser [8].

**Theorem 3** Let $p + 1$ be the order of a skew-Hadamard matrix or conference matrix. Suppose $p = q^2 + q + 1$ for some prime power $q$. Then there exists a self-orthogonal code over $\mathbb{Z}_k$ of length $2p$, with $m$ independent rows in its generator matrix and minimum distance $d$ whenever $p + q = (q + 1)^2 \equiv 0 \pmod k$.

**Proof.** Write the skew-Hadamard matrix $S + I$, minus its diagonal entries, or conference matrix as

$$
\begin{bmatrix}
0 & e \\
\pm e^T & P
\end{bmatrix}
$$

where $e$ is the $1 \times p$ matrix of ones. Then $P$ is a $p \times p$ matrix satisfying

$$
PP^T = pI - J.
$$

Write $Q$ for an incidence matrix of the projective plane over $GF(q)$. Then $Q$, of order $p = q^2 + q + 1$, is circulant and satisfies

$$
QQ^T = qI + J.
$$

Now $G_1 = [P \ Q]$ generates the required self-orthogonal code over $\mathbb{Z}_k$ of length $2p$ and with $m$, $m \leq p$ independent rows in its generator matrix as $G_1G_1^T = (p + q)I = (q + 1)^2I \equiv 0$. $\Box$

**Corollary 2** Let $p + 1$ be the order of a skew-Hadamard matrix or conference matrix. Suppose $p = q^2 + q + 1$ for some prime power $q$, and $q \equiv 2 \pmod 3$. Then there exists a self-orthogonal $[2p, m, d]$ ternary code with $m \leq p - 1$. Note that $m = p$ iff $q \equiv 1 \pmod 3$ and thus $G_1 = [P \ Q]$ is the generator matrix of a self-dual code.
Proof.  Use theorem 3. \hfill \Box

Example 3 Let $q = 2$, $p = 7$, $P = \text{circ}(0, 1, 1, -1, 1, -1, -1)$ and $Q = \text{circ}(1, 1, 0, 1, 0, 0, 0)$. We consider the matrix $[P \ Q]$ and we remove its first row. Then the derived matrix is the generator matrix of a $[14, 6, 6; 3]$ code with weight enumerator

$$W(z) = 1 + 84z^6 + 476z^9 + 168z^{12}.$$ 

Theorem 4 The codes over $GF(3)$ and $\mathbb{Z}_4$ we obtain using $G_1$ are

(i) $[2p, p, d]$ for $q \equiv 1 \pmod{3}$

(ii) $[2p, p - 1, d]$ for $q \equiv 0, 2 \pmod{3}$ and $q \equiv 0, 1, 2, 3 \pmod{4}$.

Proof. Consider the matrix $P$ of order $p = q^2 + q + 1$. Now $PP^T = (q^2 + q + 1)I - J$ and det $PP^T \equiv 0 \pmod{3}$ and $0 \pmod{4}$. Now consider $P'$ with one row of $P$ removed. Then the matrix $P'$ has size $(q^2 + q) \times (q^2 + q + 1)$ and so $P'P'^T$ is of order $q^2 + q$ and has the following form:

$$P'P'^T = \begin{bmatrix} q^2 + q & -1 & -1 & \cdots & -1 \\ -1 & q^2 + q & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & q^2 + q \end{bmatrix}$$

and det $P'P'^T = (1)(q^2 + q + 1)^{q^2 + q - 1} \neq 0$ for $q \equiv 0, 2 \pmod{3}$ and $q \equiv 0, 1, 2, 3 \pmod{4}$. Hence the rank of the matrix $P'$ is $p - 1$ for these cases.

Now the matrix $Q$ satisfies $QQ^T = qI + J$ and det $QQ^T = (q + 1)^2(q^2 + q) \neq 0 \pmod{3}$ for $q \equiv 1 \pmod{3}$. Hence the rank of the matrix $Q$ is $p$ for this case. \hfill \Box

Remark 1 We recall that a self-orthogonal code, $C$, of length $2p$, with $p$ independed rows in its generator matrix and distance $d_1$ with $C^\perp$ a self-orthogonal code of length $2p$ and $p$ independed rows in its generator matrix with distance $d_2$ we have that $C = C^T$ and so $C$ is in fact self-dual.

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**Theorem 5** Let \( p + 1 \) be the order of a skew-Hadamard matrix or a conference matrix. Suppose \( p = q^2 + q + 1 \) for some prime power \( q \). Then there exists a self-orthogonal \( \mathbb{Z}_k \)-code of length \( 2p \), with \( m \) independent rows in its generator matrix and minimum distance \( d \), whenever \( p + q \equiv 0 \pmod{k} \).

**Proof.** Construct the matrices \( P \) and \( Q \) as in the proof of theorem 3. Set
\[
G_3 = \begin{bmatrix}
P & Q \\
Q^T & -P^T
\end{bmatrix}
\]
We have that
\[
G_3G_3^T = \begin{bmatrix}
P & Q \\
Q^T & -P^T
\end{bmatrix} \begin{bmatrix}
P^T & Q \\
Q^T & -P
\end{bmatrix} = \begin{bmatrix}
PP^T + QQ^T & PQ - QP \\
Q^TP^T - P^TQ & Q^TQ + P^TP
\end{bmatrix}
\]
If \( PQ = QP \) (for example, this is true if \( P \) is circulant, in which case \( p \) is prime) then this matrix generates the required self-orthogonal code of length \( 2p \) with \( m \) independent rows in its generator matrix, as \( G_3G_3^T = (q + 1)^2I_m \equiv 0 \pmod{k} \).

\( \square \)

**Theorem 6** Let \( p + 1 \) be the order of a skew-Hadamard matrix or a conference matrix. Suppose \( p = q^2 + q + 1 \) for some prime power \( q \). Then there exists a self-dual \( \mathbb{Z}_k \)-code of length \( 4p \), with \( 2p \) independent rows in its generator matrix and minimum distance \( d \), whenever \( p + q + a \equiv 0 \pmod{k} \) for some \( a \in \mathbb{Z}_k \).

**Proof.** Construct the matrices \( P, Q \) and \( G_3 \) as in the proof of theorem 5. Set \( G_4 = [I_{2p}, G_3] \). If \( PQ = QP \) (for example, this is true if \( P \) is circulant, in which case \( p \) is prime) then the matrix \( G_4 \) generates the required self-dual code of length \( 4p \) with \( 2p \) independent rows in its generator matrix, as \( G_4G_4^T = (q + p + a)I_{2p} \).

\( \square \)

We are able to use the considerable literature on the minimum distance of codes generated by skew-Hadamard matrices, \( I + S \), minus its diagonal entries, to obtain lower bounds for the minimum distance of codes with generator matrix \( [P \; Q] \), where \( P \) and \( Q \) are given in the proof of Theorem 3 via the following lemma:
Lemma 1 Suppose $A$ and $B$ are two matrices of order $n$ with elements from $\mathbb{Z}_k$ and $\det(A) \neq 0$. We denote the minimum weights among all linear combinations of their rows (over $\mathbb{Z}_k$) by $d_A$ and $d_B$ respectively. Then the code, $C$, with generator matrix $[A \ B]$ has minimum Hamming distance $d_C \geq d_A + d_B$.

Remark 2 There are many pairs $(p, q)$ which satisfy the conditions of Theorem 3. The first few pairs are $(7, 2), (13, 3), (31, 5), (73, 8), (91, 9), (183, 13), (307, 17), (757, 27), (1723, 41)$.

Example 4 1. Let $q = 3, p = 13, P = circ(0, 1, -1, 1, -1, -1, -1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ and $Q = circ(1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. We consider the matrix $[P \ Q]$ and we remove its first row. Then the derived matrix is the generator matrix of a self-orthogonal $\mathbb{Z}_4$-code of length 26, with 12 independent rows in its generator matrix and minimum distance 8 with weight enumerator

$$W(z) = 1 + 390z^8 + 1716z^{10} + 40902z^{12} + 17056z^{13} + 226720z^{14} + 422656z^{15} + 541593z^{16} + 2348320z^{17} + 1012440z^{18} + 4010240z^{19} + 2425436z^{20} + 2384296z^{21} + 2247648z^{22} + 59194z^{23} + 472680z^{24} + 56160z^{25} + 10868z^{26}.$$ 

2. Let $q = 5, p = 31, P = circ(0, -1, 1, -1, 1, -1, 1, -1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ and $Q = circ(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. We consider the matrix $[P \ Q]$ and we remove its first row. Then the derived matrix is the generator matrix of a self-orthogonal code over $GF(3)$ of length 62, with 30 independent rows in its generator matrix and minimum distance 12. Thus we can obtain a $[62, m, d; 3]$ code for all $m \leq 30$ and with $d(m) \geq 12$ by removing rows.

References


