On Full Orthogonal Designs in Order 56

S. Georgiou; C. Koukouvinos; and Jennifer Seberry

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Abstract

We find new full orthogonal designs in order 56 and show that of 1285 possible
\( OD(56, s_1, s_2, s_3, 56 - s_1 - s_2 - s_3) \) 163 are known, of 261 possible \( OD(56, s_1, s_2, 56 - s_1 - s_2) \)
179 are known. All possible \( OD(56, s_1, 56 - s_1) \) are known.

Key words and phrases: Construction, sequences, circulant matrices, amicable sets, orthogonal
designs.

AMS Subject Classification: Primary 05B15, 05B20, Secondary 62K05.

1 Introduction

An orthogonal design of order \( n \) and type \( (s_1, s_2, \ldots, s_u) \) \( (s_i > 0) \), denoted \( OD(n, s_1, s_2, \ldots, s_u) \),
on the commuting variables \( x_1, x_2, \ldots, x_u \) is an \( n \times n \) matrix \( A \) with entries from \( \{0, \pm x_1, \pm x_2, \ldots, \pm x_u \} \) such that

\[
AA^T = \left( \sum_{i=1}^{u} s_i x_i^2 \right) I_n
\]

Alternatively, the rows of \( A \) are formally orthogonal and each row has precisely \( s_i \) entries of
the type \( \pm x_i \). In [2], where this was first defined, it was mentioned that

\[
A^T A = \left( \sum_{i=1}^{u} s_i x_i^2 \right) I_n
\]

and so our alternative description of \( A \) applies equally well to the columns of \( A \). It was also
shown in [2] that \( u \leq \rho(n) \), where \( \rho(n) \) (Radon’s function) is defined by \( \rho(n) = 8c + 2^d \), when
\( n = 2^a b, \ b \ \text{odd}, \ a = 4c + d, \ 0 \leq d < 4 \).

A weighing matrix \( W = W(n, k) \) is a square matrix with entries 0, \pm 1 having \( k \) non-zero entries per row and
column and inner product of distinct rows zero. Hence \( W \) satisfies \( WW^T = kI_n \), and \( W \) is equivalent to an orthogonal
design \( OD(n; k) \). The number \( k \) is called the weight of \( W \). If \( k = n \), that is, all the entries of \( W \) are \pm 1 and \( WW^T = nI_n \), then \( W \) is
called an Hadamard matrix of order \( n \). In this case \( n = 1, 2 \) or \( n \equiv 0 (\text{mod} 4) \).

*Department of Mathematics, National Technical University of Athens, Zografou 15773, Athens, Greece.
†Department of Computer Science, University of Wollongong, Wollongong, NSW, 2522, Australia.
Given the sequence \( A = \{a_1, a_2, \ldots, a_n\} \) of length \( n \) the non-periodic autocorrelation function \( N_A(s) \) is defined as

\[
N_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \quad s = 0, 1, \ldots, n - 1,
\]  
(1)

If \( A(z) = a_1 + a_2 z + \ldots + a_n z^{n-1} \) is the associated polynomial of the sequence \( A \), then

\[
A(z)A(z^{-1}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j z^{i-j} = N_A(0) + \sum_{s=1}^{n-1} N_A(s) (z^s + z^{-s}), \quad z \neq 0.
\]  
(2)

Given \( A \) as above of length \( n \) the periodic autocorrelation function \( P_A(s) \) is defined, reducing \( i + s \) modulo \( n \), as

\[
P_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \quad s = 0, 1, \ldots, n - 1.
\]  
(3)

The following theorem which uses four circulant matrices in the Goethals-Seidel array is very useful in our construction for orthogonal designs.

**Theorem 1** [3, Theorem 4.49] Suppose there exist four circulant matrices \( A, B, C, D \) of order \( n \) satisfying

\[
AA^T + BB^T + CC^T + DD^T = f I_n
\]

Let \( R \) be the back diagonal matrix. Then

\[
GS = \begin{pmatrix}
A & BR & CR & DR \\
-BR & A & D^T R & -C^T R \\
-CR & -D^T R & A & B^T R \\
-DR & C^T R & -B^T R & A
\end{pmatrix}
\]

is a \( W(4n, f) \) when \( A, B, C, D \) are \( (0, 1, -1) \) matrices, and an orthogonal design \( OD(4n; s_1, s_2, \ldots, s_u) \) on \( x_1, x_2, \ldots, x_u \) when \( A, B, C, D \) have entries from \( \{0, \pm x_1, \ldots, \pm x_u\} \) and \( f = \sum_{j=1}^{u} (s_j x_j^2) \).

**Corollary 1** If there are four sequences \( A, B, C, D \) of length \( n \) with entries from \( \{0, \pm x_1, \pm x_2, \pm x_3, \pm x_4\} \) with zero periodic or non-periodic autocorrelation function, then these sequences can be used as the first rows of circulant matrices which can be used in the Goethals-Seidel array to form an \( OD(4n; s_1, s_2, s_3, s_4) \). We note that if the non-periodic autocorrelation function is zero, then there are sequences of length \( n + m \) for all \( m \geq 0 \).

This method for constructing orthogonal designs was used in [1, 5].

Throughout this paper we will use the definition and notation of Koukouvinos, Mitrouli, Seberry and Karabalas [5].

A pair of matrices \( A, B \) is said to be amicable (anti-amicable) if \( AB^T - BA^T = 0 \) (\( AB^T + BA^T = 0 \)). Following [7] a set \( \{A_1, A_2, \ldots, A_{2n}\} \) of square real matrices is said to be amicable if

\[
\sum_{i=1}^{n} \left( A_{\sigma(2i-1)} A_{\sigma(2i)}^T - A_{\sigma(2i)} A_{\sigma(2i-1)}^T \right) = 0
\]  
(4)
for some permutation $\sigma$ of the set $\{1, 2, \ldots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. So

$$\sum_{i=1}^{n} \left( A_{2i-1} A_{2i}^{T} - A_{2i} A_{2i-1}^{T} \right) = 0. \tag{5}$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout this paper $R_k$ denotes the back diagonal identity matrix of order $k$.

Let $\{A_i\}_{i=1}^{s}$ be an amicable set of circulant matrices of order $t$, satisfying the additive property for $(s_1, s_2, \ldots, s_k)$. Then the Kharaghani array

$$H = \begin{pmatrix}
A_1 & A_2 & A_4 R_n & A_5 R_n & A_6 R_n & A_7 R_n & A_8 R_n & A_9 R_n & A_{10} R_n \\
-A_2 & A_1 & -A_4 R_n & -A_5 R_n & -A_6 R_n & -A_7 R_n & -A_8 R_n & -A_9 R_n & -A_{10} R_n \\
-A_4 R_n & -A_5 R_n & A_1 & A_2 & -A_6 R_n & A_7 R_n & -A_8 R_n & -A_9 R_n & -A_{10} R_n \\
-A_5 R_n & A_4 R_n & -A_2 & A_1 & A_6 R_n & -A_7 R_n & -A_8 R_n & -A_9 R_n & -A_{10} R_n \\
-A_6 R_n & A_5 R_n & A_4 R_n & -A_3 R_n & A_1 & A_2 & -A_9 R_n & -A_{10} R_n & -A_{10} R_n \\
-A_7 R_n & A_6 R_n & A_5 R_n & -A_4 R_n & -A_2 & A_1 & A_3 R_n & A_4 R_n & A_5 R_n \\
-A_8 R_n & A_7 R_n & A_6 R_n & A_5 R_n & A_4 R_n & -A_3 R_n & A_2 & A_1 & A_6 R_n \\
-A_9 R_n & A_8 R_n & A_7 R_n & A_6 R_n & A_5 R_n & -A_4 R_n & -A_3 R_n & -A_2 & A_1
\end{pmatrix}$$

is an $OD(8t; s_1, s_2, \ldots, s_k)$.

The Kharaghani array which uses amicable sets of eight matrices is also very useful in our constructions for orthogonal designs.

The following lemma applies a lemma given in Georgiou, Koukouvinos, Mitrouli and Seberry [1] to determine the number of possible tuples to be searched determining the size of search space for orthogonal designs in order 56.

**Lemma 1** Let $n = 4m = 56$ be the order of an orthogonal design then the number of cases which must be studied to determine whether all orthogonal designs exist is

1. $\frac{1}{16} n^2 = 784$ when 2-tuples are considered;
2. $\frac{1}{12} (2n^2 + 7n + 6) = 5004$ when 3-tuples are considered;
3. $\frac{1}{576} (n^4 + 6n^3 - 2n^2 - 24n + 64) = 18890$ when 4-tuples are considered.

2 New full orthogonal designs from smaller orders

**Theorem 2** There are $OD(56; s_1, s_2, 65-s_1, 56-s_1)$ constructed using the full $OD(28; s_1, 28-s_1)$ given in [2, 5, 6] for:

$$(1, 1, 27, 27) \quad (5, 5, 23, 23) \quad (9, 9, 19, 19) \quad (13, 13, 15, 15)$$
$$(2, 2, 26, 26) \quad (6, 6, 22, 22) \quad (10, 10, 18, 18) \quad (14, 14, 14, 14)$$
$$(3, 3, 25, 25) \quad (7, 7, 21, 21) \quad (11, 11, 17, 17)$$
$$(4, 4, 24, 24) \quad (8, 8, 20, 20) \quad (12, 12, 16, 16)$$

**Proof.** We use the amicable orthogonal designs of type $AOD(2; (1, 1), (1, 1))$ in order two with the two variable designs in order 28 to obtain the desired designs in order 56. \[\square\]
Theorem 3 There are full OD($56; s_1, s_2, s_3, 56 - s_1 - s_2 - s_3$) constructed using the full OD($28; s_1, s_2, 28 - s_1 - s_2$) and OD($28; s_1, s_2, s_3, 28 - s_1 - s_2 - s_3$) designs in order 28 for the 4-tuples given in Table 2.

\[
\begin{array}{cccc}
(1, 1, 2, 52) & (2, 2, 13, 39) & (2, 13, 13, 28) & (4, 8, 22, 22) \\
(1, 1, 4, 50) & (2, 2, 14, 38) & (2, 13, 15, 25) & (4, 9, 9, 34) \\
(1, 1, 6, 48) & (2, 2, 16, 36) & (2, 14, 14, 26) & (4, 12, 20, 20) \\
(1, 1, 12, 42) & (2, 2, 18, 34) & (2, 16, 18, 20) & (4, 13, 13, 26) \\
(1, 1, 16, 38) & (2, 2, 25, 27) & (2, 16, 19, 19) & (4, 14, 19, 19) \\
(1, 1, 18, 36) & (2, 2, 26, 26) & (2, 18, 18, 18) & (4, 16, 18, 18) \\
(1, 1, 25, 28) & (2, 3, 3, 48) & (3, 3, 12, 38) & (4, 17, 17, 18) \\
(1, 2, 2, 51) & (2, 3, 12, 39) & (3, 3, 14, 36) & (5, 5, 10, 36) \\
(1, 2, 3, 50) & (2, 3, 15, 36) & (3, 3, 20, 30) & (5, 5, 18, 28) \\
(1, 2, 16, 37) & (2, 4, 25, 25) & (3, 5, 12, 36) & (5, 10, 18, 23) \\
(1, 2, 17, 36) & (2, 6, 12, 42) & (4, 4, 4, 44) & (5, 15, 18, 18) \\
(1, 2, 25, 27) & (2, 6, 12, 36) & (4, 4, 8, 40) & (6, 6, 6, 38) \\
(1, 3, 16, 35) & (2, 6, 18, 30) & (4, 4, 12, 36) & (6, 6, 8, 36) \\
(1, 3, 25, 26) & (2, 6, 24, 24) & (4, 4, 16, 32) & (6, 7, 7, 36) \\
(1, 6, 12, 37) & (2, 8, 8, 38) & (4, 4, 20, 28) & (6, 10, 10, 30) \\
(1, 6, 13, 36) & (2, 8, 10, 36) & (4, 7, 7, 38) & (6, 12, 18, 20) \\
(1, 7, 12, 36) & (2, 9, 9, 36) & (4, 8, 8, 36) & (6, 12, 19, 19) \\
(1, 8, 18, 19) & (2, 9, 18, 27) & (4, 8, 12, 32) & (6, 14, 18, 18) \\
(2, 2, 2, 50) & (2, 12, 18, 24) & (4, 8, 18, 25) & (6, 15, 15, 20) \\
(2, 2, 8, 44) & (2, 12, 21, 21) & (4, 8, 20, 24) & (7, 7, 14, 28) \\
\end{array}
\]

Table 1: Full 4-variable OD($56; s_1, s_2, s_3, 56 - s_1 - s_2 - s_3$) constructed from full three and four variable designs in order 28.

Theorem 4 There are OD($56; s_1, s_1, 2s_2, 2s_3, 56 - 2s_1 - 2s_2 - 2s_3$) constructed using the Multiplication Theorem [3, Lemma 4.11] with the full OD($28; s_1, s_2, s_3, 28 - s_1 - s_2 - s_3$) given in [2, 5, 6] for the values given in Table 2.

\[
\begin{array}{cccc}
(1, 1, 2, 2, 50) & (2, 2, 8, 8, 36) & (2, 9, 9, 18, 18) & (7, 7, 14, 14, 14) \\
(1, 1, 2, 16, 36) & (2, 2, 13, 13, 26) & (4, 4, 4, 4, 36) & (8, 8, 8, 16, 16) \\
(1, 1, 2, 26, 26) & (2, 2, 16, 18, 18) & (4, 4, 8, 8, 32) & (8, 8, 10, 10, 20) \\
(1, 1, 6, 12, 36) & (2, 3, 3, 12, 36) & (4, 4, 8, 20, 20) & (9, 9, 10, 10, 18) \\
(1, 1, 18, 18, 18) & (2, 6, 6, 6, 36) & (4, 8, 8, 18, 18) & (5, 5, 10, 18, 18) \\
(2, 2, 2, 25, 25) & (2, 6, 12, 18, 18) & (5, 5, 10, 18, 18) & (7, 7, 14, 28) \\
\end{array}
\]

Table 2: Full 5-variable designs in order 56 from full 4-variable designs in order 28.

In table 3 we present the new amicable sets of eight matrices which can be used in the Kharaghani array to construct some new full orthogonal designs in order 56.
<table>
<thead>
<tr>
<th>Type</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
<th>$A_7$</th>
<th>$A_8$</th>
<th>ZERO</th>
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<tr>
<td></td>
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<td>$b, d, d, d, d, d, d$</td>
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<td>n=7</td>
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<tr>
<td></td>
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<td>n=7</td>
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<tr>
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<td>$c, d, d, d, d, d, d$</td>
<td>$\bar{a}, b, b, a, a, a$</td>
<td>PAF</td>
<td>n=7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: New full orthogonal designs in order 56 constructed from new amicable sets of eight matrices.
<table>
<thead>
<tr>
<th>Type</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
<th>$A_7$</th>
<th>$A_8$</th>
<th>ZERO</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,14,14,27)</td>
<td>$a, d, d, d, d, d, d$</td>
<td>$d, a, a, a, a, a$</td>
<td>$b, d, d, d, d, d, d$</td>
<td>$d, b, b, b, b, b$</td>
<td>$c, d, d, d, d, d, d$</td>
<td>$d, a, a, a, a, a$</td>
<td>$b, b, a, b, b, b$</td>
<td>$b, a, b, a, a, a$</td>
<td>PAF n=7</td>
</tr>
<tr>
<td>(1,20,35)</td>
<td>$a, a, a, a, a, a$</td>
<td>$d, a, a, a, a, a$</td>
<td>$b, b, a, b, a, a$</td>
<td>$d, b, a, b, a, a$</td>
<td>$b, b, a, b, b, a$</td>
<td>$a, a, a, a, a, a$</td>
<td>$b, b, a, b, a, a$</td>
<td>$b, b, a, b, b, b$</td>
<td>PAF n=7</td>
</tr>
<tr>
<td>(2,2,8,8,18,18)</td>
<td>$a, b, b, a, b, a, a$</td>
<td>$d, b, a, a, a, a$</td>
<td>$a, b, b, a, a, a$</td>
<td>$d, b, b, b, b, b$</td>
<td>$b, b, b, b, b, b$</td>
<td>$a, a, a, a, a, a$</td>
<td>$b, b, b, b, b, b$</td>
<td>$a, a, a, a, a, a$</td>
<td>PAF n=7</td>
</tr>
<tr>
<td>(2,4,22,28)</td>
<td>$a, a, a, a, a, a, a$</td>
<td>$d, b, b, b, b, b, b$</td>
<td>$a, b, b, b, b, b, b$</td>
<td>$d, b, b, b, b, b, b$</td>
<td>$a, b, b, b, b, b, b$</td>
<td>$a, b, b, b, b, b, b$</td>
<td>$h, h, h, h, h, h$</td>
<td>$h, h, h, h, h, h$</td>
<td>PAF n=7</td>
</tr>
<tr>
<td>(3,22,31)</td>
<td>$a, a, a, a, a, a, a$</td>
<td>$d, b, b, b, b, b, b$</td>
<td>$a, b, b, b, b, b, b$</td>
<td>$d, b, b, b, b, b, b$</td>
<td>$a, b, b, b, b, b, b$</td>
<td>$a, b, b, b, b, b, b$</td>
<td>$h, h, h, h, h, h$</td>
<td>$h, h, h, h, h, h$</td>
<td>PAF n=7</td>
</tr>
<tr>
<td>(4,4,4,10,10,10,10)</td>
<td>$b, c, a, c, d, d, d, d$</td>
<td>$f, g, e, g, h, h, h$</td>
<td>$b, c, a, c, d, d, d, d$</td>
<td>$f, g, e, g, h, h, h$</td>
<td>$b, d, a, d, d, d, d, d$</td>
<td>$h, h, e, h, g, g, g$</td>
<td>$b, d, a, d, d, d, d, d$</td>
<td>$b, d, a, d, d, d, d, d$</td>
<td>NPAF n=7</td>
</tr>
<tr>
<td>(4,6,46)</td>
<td>$c, e, e, c, c, b, a$</td>
<td>$e, c, e, c, a, b, c$</td>
<td>$c, e, e, c, c, b, c$</td>
<td>$e, c, c, c, a, b, c$</td>
<td>$c, e, c, c, c, e, c$</td>
<td>$c, e, e, c, c, c, c$</td>
<td>$c, e, c, c, c, e, c$</td>
<td>$c, e, e, c, c, c, c$</td>
<td>PAF n=7</td>
</tr>
<tr>
<td>(4,7,21,24)</td>
<td>$a, a, a, a, a, a, d$</td>
<td>$f, f, f, f, f, e, f, f$</td>
<td>$f, f, f, f, f, e, f, f$</td>
<td>$f, f, f, f, f, e, f, f$</td>
<td>$f, f, f, f, f, e, f, f$</td>
<td>$f, f, f, f, f, e, f, f$</td>
<td>$f, f, f, f, f, e, f, f$</td>
<td>$f, f, f, f, f, e, f, f$</td>
<td>NPAF n=7</td>
</tr>
<tr>
<td>(7,7,7,7,7,7,7,7)</td>
<td>$b, b, b, b, b, b, b, b$</td>
<td>$f, f, f, f, f, f, f, f$</td>
<td>$f, f, f, f, f, f, f, f$</td>
<td>$f, f, f, f, f, f, f, f$</td>
<td>$f, f, f, f, f, f, f, f$</td>
<td>$f, f, f, f, f, f, f, f$</td>
<td>$f, f, f, f, f, f, f, f$</td>
<td>$f, f, f, f, f, f, f, f$</td>
<td>NPAF n=7</td>
</tr>
</tbody>
</table>

Table 3 (cont.)
<table>
<thead>
<tr>
<th>Type</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
<th>$A_7$</th>
<th>$A_8$</th>
<th>$\text{ZERO}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(7,7,18,24)</td>
<td>$(a, a, a, a, a, a, d)$</td>
<td>$(b, b, b, b, b, b, b)$</td>
<td>$(a, a, a, a, d, a, d)$</td>
<td>$(\bar{a}, a, a, a, d, a, d)$</td>
<td>$(\bar{a}, a, a, a, d, a, d)$</td>
<td>$(\bar{a}, a, a, a, d, a, d)$</td>
<td>$(\bar{a}, a, a, a, d, a, d)$</td>
<td>NPAF n=7</td>
<td></td>
</tr>
<tr>
<td>(8,11,37)</td>
<td>$(\bar{a}, b, b, a, a, a, a)$</td>
<td>$(c, b, b, b, b, b, b)$</td>
<td>$(\bar{a}, b, b, b, b, b, b)$</td>
<td>$(\bar{a}, b, b, b, b, b, b)$</td>
<td>$(\bar{a}, b, b, b, b, b, b)$</td>
<td>$(\bar{a}, b, b, b, b, b, b)$</td>
<td>$(\bar{a}, b, b, b, b, b, b)$</td>
<td>PAF n=7</td>
<td></td>
</tr>
<tr>
<td>(11,14,31)</td>
<td>$(\bar{a}, b, b, a, a, a, a)$</td>
<td>$(c, a, a, a, a, a, a)$</td>
<td>$(\bar{a}, b, b, b, b, b, b)$</td>
<td>$(\bar{a}, b, b, b, b, b, b)$</td>
<td>$(\bar{a}, b, b, b, b, b, b)$</td>
<td>$(\bar{a}, b, b, b, b, b, b)$</td>
<td>$(\bar{a}, b, b, b, b, b, b)$</td>
<td>PAF n=7</td>
<td></td>
</tr>
</tbody>
</table>

**Remark 1** We note that amicable sets of eight matrices of type $(4,4,4,10,10,10,10)$ and $(7,7,7,7,7,7,7)$ which are used for constructing OD's in order 56 are also found in [4].

<table>
<thead>
<tr>
<th>(1,2,3,30)</th>
<th>(1,2,16,25)</th>
<th>(2,18,18,18)</th>
<th>(4,4,20,28)</th>
<th>(4,12,20,20)</th>
<th>(8,10,14,24)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2,25,28)</td>
<td>(2,2,8,44)</td>
<td>(3,3,25,25)</td>
<td>(4,4,24,24)</td>
<td>(4,13,14,25)</td>
<td>(8,10,18,20)</td>
</tr>
<tr>
<td>(1,3,8,44)</td>
<td>(2,2,16,36)</td>
<td>(3,8,19,25)</td>
<td>(4,8,36,34)</td>
<td>(4,14,14,21)</td>
<td>(8,12,18,18)</td>
</tr>
<tr>
<td>(1,3,13,39)</td>
<td>(2,2,18,34)</td>
<td>(3,8,20,25)</td>
<td>(4,8,10,34)</td>
<td>(4,14,18,20)</td>
<td>(8,14,20,20)</td>
</tr>
<tr>
<td>(1,3,14,38)</td>
<td>(2,2,26,26)</td>
<td>(3,9,19,25)</td>
<td>(4,8,14,30)</td>
<td>(4,16,18,18)</td>
<td>(10,10,26,24)</td>
</tr>
<tr>
<td>(1,3,19,33)</td>
<td>(2,3,25,26)</td>
<td>(3,13,14,25)</td>
<td>(4,8,18,26)</td>
<td>(7,7,7,25)</td>
<td>(10,10,26,24)</td>
</tr>
<tr>
<td>(1,5,25,25)</td>
<td>(2,8,38,38)</td>
<td>(3,14,14,25)</td>
<td>(4,8,20,24)</td>
<td>(7,7,21,21)</td>
<td>(10,10,16,20)</td>
</tr>
<tr>
<td>(1,8,19,28)</td>
<td>(2,8,10,36)</td>
<td>(4,4,4,44)</td>
<td>(4,10,10,32)</td>
<td>(7,14,14,21)</td>
<td>(10,10,18,18)</td>
</tr>
<tr>
<td>(1,8,22,25)</td>
<td>(2,8,18,28)</td>
<td>(4,4,8,40)</td>
<td>(4,10,12,30)</td>
<td>(8,8,10,30)</td>
<td>(10,12,14,20)</td>
</tr>
<tr>
<td>(1,11,19,25)</td>
<td>(2,8,20,26)</td>
<td>(4,4,10,38)</td>
<td>(4,10,14,28)</td>
<td>(8,8,18,22)</td>
<td>(10,14,18,18)</td>
</tr>
<tr>
<td>(1,13,14,28)</td>
<td>(2,10,18,26)</td>
<td>(4,4,14,34)</td>
<td>(4,10,18,24)</td>
<td>(8,8,20,20)</td>
<td>(14,14,14,14)</td>
</tr>
<tr>
<td>(1,13,17,25)</td>
<td>(2,16,18,20)</td>
<td>(4,4,18,30)</td>
<td>(4,10,20,22)</td>
<td>(8,10,10,28)</td>
<td></td>
</tr>
</tbody>
</table>

**Table 4:** Full 4-variable $OD(56; s_1, s_2, s_3, 56 - s_1 - s_2 - s_3)$ constructed from full designs presented in table 2.

### 3 Full designs with even parameters

We note that Seberry [8] showed that if all $OD(n; x, y, n-x-y)$ exist then all $OD(2n; z, w, 2n-z-w)$ exist for $s \geq 0$ an integer. In particular if all $OD(2^p x, y, 2^p - x-y)$ exist, for some odd integer $p$, then all $OD(2^{p+s}p; z, w, 2^{p+s} - z-w)$ exist for $s \geq 0$ an integer we observe

**Lemma 2** If all $OD(2^p; 2x, 2y, 2p-2x-2y)$ exist, for some odd integer $p$, then all $OD(2^{p+s}; 2z, 2w, 2^{p+s} - 2z - 2w)$ exist for $s \geq 0$ an integer.
Table 5: The existence of these 82 full $OD(56; s_1, s_2, 72 - s_1 - s_2)$ is not yet established.

**Corollary 2** If $OD(56; 6, 16, 34)$ exist then all $OD(2^{s+37}; 2z, 2w, 2^{s+37} - 2z - 2w)$ exist for $s \geq 0$ an integer.

**Proof.** A search of full $OD(56; x, y, 56 - x - y)$ show only the parameters indicated are as yet unsolved. \[\square\]

4 Summary

We have found new designs in order 56 and shown that of 1285 possible $OD(56; s_1, s_2, s_3, 56 - s_1 - s_2 - s_3)$ 163 are known; of 261 possible $OD(56; s_1, s_2, 56 - s_1 - s_2)$ 179 are known; and all possible $OD(56; s_1, 56 - s_1)$ are known.

References


