On $G$-Matrices

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Abstract
$G$-matrices for the new orders 21, 23, 25 and 27 are constructed. Some constructions for Hadamard matrices and orthogonal designs using $G$-matrices are also presented.

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1 Introduction
Let $X_1, X_2, X_3, X_4$, be four type 1 ± 1 matrices on the same group of order $n$ (odd) with the properties:

(i) $(X_i - I)^T = -X_i + I$, $i = 1, 2$,

(ii) $X_i^T = X_i$, $i = 3, 4$ and the diagonal elements are positive,

(iii) $X_i X_j = X_j X_i$,

(iv) $X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = 4n I_n$.

Call such matrices $G$-matrices. These were first introduced and applied to construct Hadamard matrices by Jennifer Seberry Wallis [3]. $G$-matrices of orders 3, 5, 7, 9, 13, 15, 19 were known previously, see [8, 10, 6]. This note constructs $G$-matrices of order 21, 23, 25 and 27 for the first time.

Remark 1 Multiplying both sides of (iv) by $J$ shows $G$-matrices can only exist for orders $n$ for which

$$4n = a^2 + b^2 + a^2 + b^2$$

where $a, b$ are odd integers. So, for example, they cannot exist for the following orders $n = 30, 11, 17, 29, 35, 39, 47$.

$G$-matrices which are constructed using four circulants exist for at least $n = 3, 5, 7, 9, 13, 15$, and 19, see [3, 5, 6], and for $n = 21, 23, 25$, and 27 which are constructed in this note. This means the first unresolved case is for $n = 31$. 

2 Construction of G-matrices

The following first rows may be used to give circulant matrices which can be used in the Goethals-Seidel array to find G-matrices of order:

\[ 4n = 4 \times 21 = 84 = 1^2 + 1^2 + 1^2 + 9^2, \]
\[ \{+++---+-++++++-+-+--+\} \]
\[ +++++-++-++++-+-+--+-+ \]
\[ ++-+++-++-+-+-+++--+++- \]
\[ +++-+-+-+-++--+-++-++-- \]

\[ 4n = 4 \times 23 = 84 = 1^2 + 1^2 + 3^2 + 9^2, \]
\[ \{++-++++---+-+-+-+--+-++--\} \]
\[ +++++---+-+-+++--+-+++-\]
\[ ++---+-+-+-+-+++--+-+++-\]
\[ +--+-+-+-+-++--+-++-++--\]
\[ +++---+-+-++++--+++-++--\]

\[ 4n = 4 \times 25 = 100 = 1^2 + 1^2 + 7^2 + 7^2, \]
\[ \{++++-+-+++-++-+-+-+-++--+-+++-\} \]
\[ +++++--+-+-+-+-++++---++--+-+++-\]
\[ ++---+-+-+-+-++--+-++-++--\]
\[ +--+-+-+-+-++--+-++-++--\]
\[ +++---+-+-++++--+++-++--\]

and \[ 4n = 4 \times 27 = 108 = 1^2 + 1^2 + 5^2 + 9^2, \]
\[ \{+++---+-+-++++--+++-++--\} \]
\[ +++-+-+-+-+-++--+-++-++--\]
\[ +--+-+-+-+-++--+-++-++--\]
\[ ++---+-+-+-+-++--+-++-++--\]
\[ +++-+-+-+-+-++--+-++-++--\]
\[ +++---+-+-++++--+++-++--\]

3 Constructions using G-matrices

First we note that we have

Lemma 1 If there exist circulant \( G \)-matrices of order \( n \) then there exists an \( OD(4n; 1, 1, 2n-1, 2n-1) \).

Corollary 1 Let \( n = 3, 5, 7, 9, 13, 15, 19, 21, 23, 25, \text{ or } 27 \). Then an \( OD(4n; 1, 1, 2n-1, 2n-1) \) exists.

We recall the following definitions and results from [3]. The following theorem shows how the Williamson construction (the \( B_2 \)) and the Goethals-Seidel construction (the \( A_2 \)) may be combined to construct Hadamard matrices.

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Theorem 1 (Jennifer Seberry Wallis (1975)) Suppose $A_i$ and $B_i$, $i = 1, 2, 3, 4$ are type 1 ($\pm 1$) matrices of order $a$ and $b$, respectively, which satisfy

(i) $A_i^T A_j = A_j^T A_i$, $i, j = 1, 2, 3, 4$
(ii) $B_i B_j^T = B_j B_i^T$, $i, j = 1, 2, 3, 4$
(iii) $\sum_{i=1}^{4} (A_i \times B_i)(A_i \times B_i)^T = 4a b I_{ab}$

then $H$ defined as

\[
\begin{pmatrix}
A_1 \times B_1 & A_2 \times B_2 & A_3 \times B_3 & A_4 \times B_4 \\
-A_1 \times B_2 & A_1 \times B_3 & A_2 \times B_4 & -A_3 \times B_1 \\
-A_1 \times B_3 & A_2 \times B_1 & A_4 \times B_1 & A_3 \times B_2 \\
-A_2 \times B_1 & -A_3 \times B_2 & A_4 \times B_2 & A_1 \times B_3
\end{pmatrix}
\]

is an Hadamard matrix of order $4ab$.

We will call the matrices $A_i \times B_i$, $i = 1, 2, 3, 4$ of the theorem $F$-matrices and we will say $H$ is a Wallis-Whiteman like Hadamard matrix.

The $A_i$ will be called the $G$-part and the $B_i$ the $W$-part of the $F$-matrix.

The following theorem shows how $G$-matrices may be used to construct $F$-matrices.

Theorem 2 (Jennifer Seberry Wallis (1975)) Let $X_1$, $X_2$, $X_3$, $X_4$ be $G$-matrices of order $a$. Suppose $A, B, C$ are suitable $\pm 1$ matrices of order $m$ for an $O(D(4n, 1), 1, 4n - 2)$ so they satisfy

(i) $AB^T$, $AC^T$, $BC^T$ are symmetric,
(ii) $AA^T + BB^T + (4n - 2)CC^T = 4nmI_m$.

Then

\[
\begin{pmatrix}
A_1 &=& I \times A + (X_1 - I) \times C \\
A_2 &=& I \times B + (X_2 - I) \times C \\
A_3 &=& X_3 \times C \\
A_4 &=& X_4 \times C
\end{pmatrix}
\]

are $F$-matrices of order mn.

Corollary 2 Let $n = 3, 5, 7, 9, 13, 15, 19, 21, 23, 25$ or 27. Suppose $A$, $B$, $C$ are pairwise antimicable ($\pm 1$) matrices of order $m$ satisfying

\[
AA^T + BB^T + (4n - 2)CC^T = 4nmI_m,
\]

Then there are $F$-matrices of order $mn$ and a Wallis-Whiteman like Hadamard matrix of order $4mn$.

Corollary 3 Let $n = 3, 5, 7, 9, 13, 15, 21, 23$. Set $A = j_{2n+1}$, $B = (I - 2j)_{2n+1}$, and $C$ the back-circulant or type 1 matrix of order $2n + 1$ obtained from the quadratic residues. Then

\[
AA^T + BB^T + (4n - 2)CC^T = 4(2n+1)I_{4n+1},
\]

and hence there are $F$-matrices of order $(2n + 1)n$ and a Wallis-Whiteman like Hadamard matrix of order $4(2n+1)n$. 

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We further extend the last theorem by observing

**Theorem 5** Let $X_1, X_2, X_3, X_4$ be $G$-matrices of order $n$. Suppose $A, B, C, D$ are suitable $\pm 1$ matrices of order $m$ for an $OD(4n; 1, 1, 2n - 1, 2n - 1)$ if they satisfy

(i) $PQ^T$ is symmetric for all $P, Q \in \{A, B, C, D\}$,

(ii) $AA^T + BB^T + (2n - 1)CC^T + (2n - 1)DD^T = 4nnI_m$.

Then defining $Y_1 = (X_1 + X_2 - 2I)/2, Y_2 = (X_1 - X_2)/2, Y_3 = (X_3 + X_4)/2$ and $Y_4 = (X_3 - X_4)/2$

$$B_1 = I \times A + Y_2 \times C + Y_2 \times D$$
$$B_2 = I \times B + Y_1 \times D + Y_4 \times C$$
$$B_3 = Y_3 \times C + Y_4 \times D$$
$$B_4 = Y_3 \times D + Y_4 \times C$$

are $F$-matrices of order $mn$.

**Corollary 4** Let $n = 3, 5, 7, 9, 13, 15, 19, 21, 23, 25$ or $27$. Suppose $A, B, C, D$ are pairwise amicable $\pm 1$ matrices of order $m$ satisfying

$$AA^T + BB^T + (2n - 1)CC^T + (2n - 1)DD^T = 4nnI_m.$$ 

Then there are $F$-matrices of order $mn$ and a Wallis-Whiteman like Hadamard matrix of order $4nn$.

**Corollary 5** Let $n = 3, 5, 7, 9, 13, 15, 19, 21, 23, 25$ or $27$. Set $A = B = J_{2n-1}$, and $C - I = D + I$ be block-covariant or type 1 matrix of order $2n - 1$ with zero diagonal obtained from the quadratic residues or the core of a symmetric conference matrix of order $2n + 2$. Then

$$AA^T + BB^T + (2n - 1)CC^T + (2n - 1)DD^T = 4(2n - 1)nI_{2n-1}.$$ 

and hence there are $F$-matrices of order $(2n - 1)n$ and a Wallis-Whiteman like Hadamard matrix of order $4(2n - 1)n$.

**References**


