On weighing matrices

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Abstract
We give new sets of \( \{0, 1, -1\} \) sequences with zero autocorrelation function, new constructions for weighing matrices and review the weighing matrix conjecture for orders \( 4t, t \in \{1, \ldots, 22\} \) establishing its validity for orders 52, 68 and 78. We give the smallest known lengths for sequences with zero autocorrelation function and weights \( \leq 100 \).

Key words and phrases: Associated polynomial, autocorrelation, construction, algorithm.
AMS Subject Classification: Primary 62K05, 62K10, Secondary 05B20

1 Introduction

Definition 1. Let \( W \) be a \((1, -1, 0)\) matrix of order \( n \) satisfying \( WW^T = kn \).
We call \( W \) a weighing matrix of order \( n \) with weight \( k \), denoted by \( W(n, k) \).

There are a number of conjectures concerning weighing matrices:

Conjecture 1. There exists a weighing matrix \( W(4t, k) \) for \( k \in \{1, \ldots, 4t\} \).

Conjecture 2. When \( n \equiv 4(\text{mod } 8) \), there exist a skew-weighing matrix (also written as an OD\((n; 1, k)\)) when \( k \leq n - 1 \), \( k = a^2 + b^2 + c^2 \), \( a, b, c \) integers except that \( n - 2 \) must be the sum of two squares.

Conjecture 3. When \( n \equiv 0(\text{mod } 8) \), there exist a skew-weighing matrix (also written as an OD\((n; 1, k)\)) for all \( k \leq n - 1 \).

The reader is referred to Geramita and Seberry [17] for all other undefined terms.

In Geramita and Seberry [17] the status of the weighing matrix conjecture is given for \( W(4t, k), k \in \{1, \ldots, 4t - 1\} \) and \( t \in \{1, \ldots, 21\} \). We give new results including resolving the conjecture in the affirmative for 52, 68 and 78. Further we give the length of the smallest \( n \) for which \( 4 - CS(n, w) \) are known for all \( w \leq 100 \).

Given the sequence \( \{a_1, a_2, \ldots, a_n\} \) of length \( n \) the non-periodic autocorrelation function \( N_A(s) \) is defined as

\[
N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1.
\]  

(1)

Utilitas Mathematica 43(1993), pp. 101-127
If \( A(z) = a_1 + a_2 z + \cdots + a_n z^{n-1} \) is the associated polynomial of the sequence \( A \), then

\[
A(z)A(z^{-1}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j z^{i-j} = N_A(z) + \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}), \quad z \neq 0. \tag{2}
\]

Given \( A \) as above of length \( n \) the periodic autocorrelation function \( P_A(s) \) is defined, reducing \( i + s \) modulo \( n \), as

\[
P_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1. \tag{3}
\]

**Notation 1** We use 0 to represent the sequence of 1 zeros, 1 for -1, and J for the matrix with every element 1. We call \( g = 2^n 10^6 26^6 \) a, b, c non-negative integers, Golay numbers. If \( A = \{a_1, a_2, \ldots, a_{n-1}, a_n\} \) is a sequence of \( n \) elements we will use \( A^* = \{a_n, a_{n-1}, \ldots, a_2, a_1\} \) to be the reverse sequence and \( \bar{A} = \{-a_1, -a_2, \ldots, -a_{n-1}, -a_n\} \) to be the sequence with all the elements negated. We use the notation \((A/B)\) for the sequence \( \{a_1, b_1, \ldots, a_n, b_n\} \) and \((A/B)\) for the sequence \( \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \).

If \( A^* = \{a_n, \ldots, a_1\} \) is the reversed sequence, then

\[
A^*(z) = z^{n-1} A(z^{-1}). \tag{4}
\]

Basso, Turyun, Golay and normal sequences are finite sequences, with zero autocorrelation function, useful in constructing orthogonal designs and Hadamard matrices \([17]\), in communications engineering \([39]\), in optics and signal transmission problems \([19, 21]\), etc.

If \( A = \{a_1, \ldots, a_n\} \) and \( B = \{b_1, \ldots, b_n\} \) are two binary \((1, -1)\) sequences of length \( n \) and

\[
N_A(z) + N_B(z) = 0 \quad \text{for} \quad s = 1, \ldots, n-1 \tag{5}
\]

then \( A, B \) are called Golay sequences of length \( n \) (abbreviated GS\( (n)\)). See \([12, 17, 19]\).

From this definition and relation (2) we conclude that two \((1, -1)\) sequences of length \( n \) are GS\( (n) \) if and only if

\[
A(z)A(z^{-1}) + B(z)B(z^{-1}) = 2n, \quad z \neq 0. \tag{6}
\]

Golay sequences GS\( (n) \) exist for \( n = 2^n 10^6 26^6 \) (Golay numbers) where \( a, b, c \) are non-negative integers and do not exist when \( n \) has any factor \( 3 \) (mod 4) \([12, 19, 20]\).

**Definition 2** \( m \) sequences \( A_1, A_2, \ldots, A_m \) of length \( n \) with entries 0, 1, -1 are called \( m \)-complementary sequences of weight \( w \), written \( m-CS(n, w) \), if

\[
N_{A_1}(s) + \ldots + N_{A_m}(s) = \begin{cases} 
0, & s = 1, \ldots, n-1 \\
\omega_n, & s = 0
\end{cases} \tag{7}
\]

which can be replaced by

\[
A_1(z)A_1(z^{-1}) + \ldots + A_m(z)A_m(z^{-1}) = \omega, \quad z \neq 0 \tag{8}
\]
Setting \( r = 1 \) in (8) we obtain
\[
\alpha^2_1 + \ldots + \alpha^2_n = w
\]
(9)

where \( \alpha_1, \ldots, \alpha_m \) are the sums of the elements of \( A_1, \ldots, A_m \) respectively.

If there exist 4-CS \((m, w, (A, (B, (C, (D\) which actually have lengths \( m_1, m_2, m_3 \) and \( m_4 \) and weight \( w \) then \((A, B, (C, D\) and \((D, (C, (A, (B\) are 4-CS \((m_1 + m_2, 2w)\).

**Example 1** \{1 0 1\} and \{1 1 -1\} are 2-CS(3,5). 
\{1 1 1 1 1 1\} and \{1 0 1 0 0 0 1\} are 2-CS(11,13). 
\{1 0 1\}, \{1 1 1\}, \{1 1 1\} are 4-CS(3,11). 
\{1 1 1 1\}, \{1 1 1 1\}, \{1 1 1 1\} are 4-CS(5,15).

We note, in the language of this paper, one very important result:

**Theorem 1** Suppose there exist 4-CS \((n, w)\). Then there exists a \(W(4n, w)\). If there exist 2-CS \((n, w)\) there exists a \(W(2n, w)\).

**Proof.** The sequences are used to make circulant matrices which are then used in the Goethals-Seidel or other appropriate array to obtain the result. ✷

**Definition 3** 2k sequences \( A_1, A_2, \ldots, A_{2k} \) of length \( n \) with entries 0, +1, −1 (where \( A_{2k} \) may be the sequence of \( n \) zeros to ensure an even number of sequences) are called \( 2k \)-disjoint complementary sequences of weight \( w \), written \( 2k - DCS(n, w) \), if they have zero nonperiodic autocorrelation function, total weight \( w \), \( A_i \perp A_{i+k} \), are also 0, +1, −1 sequences with maximum length \( n \) and zero nonperiodic autocorrelation function for \( i = 1, 3, \ldots, 2k − 1 \). These also satisfy equations (7), (8), and (9).

We note that 2-DCS \((m, w)\) always give 2-CS \((m, 2w)\).

4-DCS \((m, w)\), \( A, B, C, D \) of lengths \( m, m, n \) and \( n \), where \( m \geq n \) will be called suitable sequences and denoted \( S(m, n; w) \).

4-CS \((m, 2m + 2n)\) with lengths \( m, n, n \) and entries +1, −1, are called base sequences. Base sequences of lengths \( n + 1, n + 1, n \), \( n \) are denoted by \( BS(2n + 1) \) and we get
\[
a^2 + b^2 + c^2 + d^2 = 4n + 2
\]
(10)

where \( a, b, c, d \) are the sums of the elements of \( A, B, C, D \) respectively.

\( BS(2n + 1) \) for all decompositions of \( 4n + 2 \), \( 4n + 3 \) into four squares for \( n = 1, 2, \ldots, 24 \) are given in [2, 23, 26], \( BS(2n + 1) \) for \( n = 25, 26, 29 \) and \( n = 2^{10} \cdot 29 \) (Golay numbers) are given in Yang [46] and for 27, 28, 30 in [37]. \( BS(47, 47, 24; 71) \) are given in [25, 26], which are equivalent to base sequences 47, 47, 24, 24 with total weight 142.

**Remark 1** Since base sequences \( BS(2n + 1) \) exist for all \( n \leq 30 \) [37], they give us 4-CS \((n + 1, 4n + 2)\) and 4-UCS \((n + 1, 2n + 1)\) for each of these \( n \). Also using them as \( \{A, C\}, \{A, C\}, \{B, D\}, \{B, D\} \) gives 4-CS \((2n + 1, 8n + 4)\).

The base sequences \( m, n, n \) may be given in the form \( \{Z, W\}, \{Z, \bar{W}\}, \{X, Y\} \), with weight \( 2n \), in this case \( \{Z, W\} \), \( \{(X + Y)/2\}, \{(X - Y)/2\} \) are 4-CS \((M, w)\) where \( M = \max(|Z|, |W|, n) \). In the case of 47, 47, 24, 24 noted above the shorter sequences have lengths 21, 23, 21, 18 and give 4-CS(24,71).

The paper [25] gives 4-CS(20,59) by the same observation.
Example 2 \{0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1\} and \{0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1\} are 3-CS(6,10) and they give \{1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \} and \{0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1\} which are 2-DCS(8,5).

\{0 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1\} and \{1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1\} are 2-CS(14,26) and they give \{1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \} and \{0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1\} which are 2-DCS(14,13).

\((G+H)/2\) and \((G-H)/2\) where \(G\) and \(H\) are Golay sequences of length \(g\) are 2-DCS(g,0).

\{1 \ 1 \ 1 \ 1 \ 0 \ 1 \}, \{1 \ 1 \ 1 \ 0 \ 0 \}, \{1 \ 1 \}, \{1 \ 1\} are SS(6,2,14).

In this paper we give some special examples of \(m-CS(n,w)\). Also we give constructions, based on \(T\)-sequences and \(TW\)-sequences, which form weighing matrices.

**Definition 4** The four sequences \(X, Y, Z, W\) of length \(n\) with entries 0, 1, -1 are called \(T\)-sequences if

(i) \(|x_i| + |y_i| + |z_i| + |w_i| = 1, \ i = 1, \ldots, n\)

(ii) \(N_X(s) + N_Y(s) + N_Z(s) + N_W(s) = \left\{ \begin{array}{ll}
0, & s = 1, \ldots, n - 1 \\
\frac{n}{2}, & s = 0
\end{array} \right.\)

Yang [44] gives another name for \(T\)-sequences and calls them four-symbol \(\delta\)-codes.

He also calls the quadruple \(Q = X + Y, R = X - Y, S = Z + W, T = Z - W\) regular \(\delta\)-code of length \(n\), where \(X, Y, Z, W\) are \(T\)-sequences of length \(n\).

We now slightly extend the definition of \(T\)-sequences.

**Definition 5** Four sequences \(A, B, C, D\) of length \(l\), with elements 0, +1, -1, total number of non-zero elements (the weight) \(w\), with zero non-periodic autocorrelation function and which further satisfy \(A \pm B \pm C \pm D\) (where addition or subtraction of sequences means adding or subtracting them element by element) are also sequences of 0, +1, or -1 with zero non-periodic autocorrelation function will be called \(TW\)-sequences and denoted \(TW(l,w)\).

**Definition 6** A triple \((F;G,H)\) of sequences is said to be a set of normal sequences for length \(n\) (abbreviated as \(NS(n)\)) if the following conditions are satisfied:

(i) \(F = (f_k)\) is a \((1,-1)\) sequence of length \(n\).

(ii) \(G = (g_k)\) and \(H = (h_k)\) are sequences of length \(n\) with entries 0, 1, -1, such that \(G + H = (g_k + h_k)\) is a \((1,-1)\) sequence of length \(n\).

(iii) \(N_F(s) + N_G(s) + N_H(s) = 0, \ s = 1, \ldots, n - 1\).

We note they are also \(3-CS(n,2n)\), the quadruple \((F;0,G,H)\) gives 4-DCS(n, 2n), while the quadruple \((F;G + H, \{F,-G\}, \{F,-H\}, \{0,G-H\})\) are \(4-CS(2n;bn)\).

In [26, Theorem 1] it was shown that the sequences \(G\) and \(H\) of Definition 6 are quasi-symmetric, i.e., if \(g_k = 0\), then \(g_{n+1-k} = 0\) and also if \(h_k = 0\), then \(h_{n+1-k} = 0\). This means that \(G \pm H\) and \(G^* \pm H\) are also 1, -1 sequences.
Definition 7 The sequences $E, F, G, H$ of lengths $2m, 2m-1, 2m, 2m$ respectively, with entries 0, ±1 are called near normal sequences for length $n = 4m+1$ (abbreviated as $N(n)$) if the following conditions are satisfied

(i) $E = (X(0,1), F = Y(0,)), \quad X$ and $Y$ are (±1)-sequences of length $m$, $0 = 0_{m-1}$ the sequences of zeros of length $m-1$, and $X(0) = \{x_1, x_2, x_3, \ldots, x_{m-1}, 0, x_m\}$.

(ii) $G$ and $H$ are quasi-symmetric supplementary (0, ±1)-sequences of length $2m$, so $G + H$ is a (±1)-sequence of length $2m$ and zeros appear symmetrically in $G$ and $H \iff g_{2m+i} = 0 = h_i \iff h_{2m+i} = 0 = g_i \iff g_i = 0 = h_i \not= 0, \quad g_i \not= 0 = h_i \not= 0 = 0$ (this condition is necessarily implied by (i) and (iii)).

(iii) The sequences $E, F, G, H$ have zero nonperiodic autocorrelation sum. The sequences $X, Y, G, H$ are called Yang quasi-symmetric sequences.

All Yang’s theorems [46, 43, 44, 45, 47, 37, 27] use Yang quasi-symmetric sequences.

Also in [25] it is shown that from Golay sequences of length $n$ we can always obtain at least two sets of normal sequences $NS(n)$ and that normal sequences of length $n$ always give base sequences $BS(2n + 1)$. In fact $BS(2n + 1)$ give $NS(n)$ and vice versa.

Normal sequences do not exist for $n = 2^{2k+1} (8k+7)$, $a, b$ non-negative integers as the numbers $4^k (8k + 7)$ can only be written as the sum of four squares. In particular $n \not= 14, 30, 46, 56, 62, 78, 94, \ldots$. It is known that $NS(n)$ does not exist. Yang [46] has given these sequences for $n = 3, 5, 7, 9, 11, 13, 15, 23, 29$, and he notes they exist for $n = 2^{10} 26$ (Golay numbers).

$NS(n)$ have not been found for $n \geq 20$ (except $n = 25, 29$, and $g$, a Golay number). We can construct $T$-sequences of length $(2n+1)$, if a new set of $NS(n)$ can be found, where $t = 2n + 1$ is the length of base sequences ($BS(2n + 1)$).

Therefore we know these sequences

(i) exist for $n \in \{1, 2, \ldots, 5, 7, 8, \ldots, 13, 15, 16, 18, 19, 20, 25, 26, 29, 32, \ldots\}$;
(ii) do not exist for $n \in \{6, 14, 17, 22, 23, 30, 46, 56, 62, 78, 94, \ldots\}$;
(iii) $n \in \{24, 27, 28, 31, \ldots\}$ are undecided.

From corollary 5.24 of [37] we have $T$-sequences or $T$-matrices for all $t \leq 100$ except possibly $73, 79, 83, 89$, and $97$. Hence we have using the construction of Cooper and (Sherry) Wallis [3] that $ODN(t; t, t, t)$ exist for these $t$. Setting the first three variables equal to 1 and the fourth equal to 0 we have

Lemma 1 There exists a $W(4, 3t)$ for all $t \leq 100$ except possibly for $73, 79, 83, 89$, and $97$.

Lemma 2 If there exist $T$-sequences of length $t, T_1, T_2, T_3, T_4$ then

$\{ T_1 + T_2 + T_3, \{-T_1 + T_2 + T_3, \{-T_1 + T_2 - T_3, \{ T_2 - T_3 - T_4 \}$

are $4 - CS(3, t)$. (We note that, similarly, there will be sequences with periodic autocorrelation function zero if $T$-matrices were used.) In particular $4 - CS(t, 3)$ exist for all $t \leq 71$ [37]. Also if $TW(t; w)$ exist the same construction gives $4 - CS(t, 3w)$.

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Example 3 \( \{11^2 - 0900\}, \{0901010\}, \{0001010\}, \{0000001\} \) are \( T \)-sequences of length 7 and so

\[\{11 - 1110\}, \{- - 1101\}, \{- - 1010\}, \{0001 - 1\}\]

are the required \( 4 - \text{CS}(7,21) \).

This means there are \( 4 - \text{CS}(19,57) \), \( 4 - \text{CS}(21,63) \), \( 4 - \text{CS}(23,69) \), \( 4 - \text{CS}(25,75) \), \( 4 - \text{CS}(29,87) \), \( 4 - \text{CS}(31,93) \), \( 4 - \text{CS}(33,99) \).

\( \square \)

Remark 2 Turyan gave a construction for \( T \)-sequences using six sequences \( X, Y, Z, Z, W, W, W \) of lengths \( n, n, n, n, n, n - 1, n - 1 \). The known results are summarized in Table 1. Hence we have

Lemma 3 If there exist six sequences \( X, Y, Z, Z, W, W \), of lengths \( n, n, n, n, n, n - 1 \) and \( n = 1 \) with non-periodic auto-correlation function zero then writing

\[ U = (X + Y)/2 \text{ and } V = (X - Y)/2 \]

of length \( n \)

\[ \{ Z, W, U\}, \{ Z, W, -V\}, \{ Z, 0_{n-1}, -U + V\}, \{ 0_n, W, -U - V\} \]

are \( 4 - \text{CS}(3n - 1; 9n - 3) \).

\( \square \)

Corollary 1 \( 4 - \text{CS}(3n - 1; 9n - 3) \) exist for even \( n = 2, 4, \ldots, 24 \). In particular \( 4 - \text{CS}(11; 33) \), \( 4 - \text{CS}(17; 51) \), \( 4 - \text{CS}(23; 69) \), and \( 4 - \text{CS}(29; 87) \) exist.

<table>
<thead>
<tr>
<th>Lengths</th>
<th>Sums of Squares</th>
<th>( X, Y, Z, W ) of lengths ( n, n, n, n - 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3,2</td>
<td>( 0^2 + 0^2 + 3^2 + 1^2 )</td>
<td>( X = Y = (++.), Z = (++), W = (+) )</td>
</tr>
<tr>
<td>7,4</td>
<td>( 2^2 + 0^2 + 3^2 + 3^2 )</td>
<td>( X = (++++)), ( Y = (++++)), ( Z = (++)), ( W = (++) )</td>
</tr>
<tr>
<td>11,6</td>
<td>( 4^2 + 1^2 + 1^2 )</td>
<td>( X = (+++), Y = (+++), Z = (++) ), ( W = (++) )</td>
</tr>
<tr>
<td></td>
<td>( 2^2 + 2^2 + 5^2 + 1^2 )</td>
<td>( X = (++++)), ( Y = (+++), Z = (++) ), ( W = (++) )</td>
</tr>
<tr>
<td>13,8</td>
<td>( 6^2 + 3^2 + 3^2 + 1^2 )</td>
<td>( X = (++++++), Y = (++++++), Z = (+), W = (+) )</td>
</tr>
<tr>
<td>19,10</td>
<td>( 2^2 + 2^2 + 7^2 + 1^2 )</td>
<td>( X = (+++++++), Y = (+++++++), Z = (+), W = (+) )</td>
</tr>
<tr>
<td></td>
<td>( 0^2 + 0^2 + 7^2 + 3^2 )</td>
<td>( X = (+++++++), Y = (+++++++), Z = (+), W = (+) )</td>
</tr>
<tr>
<td>23,12</td>
<td>( 4^2 + 2^2 + 7^2 + 1^2 )</td>
<td>( X = (+++++++), Y = (+++++++), Z = (+), W = (+) )</td>
</tr>
</tbody>
</table>

Table 1: \( \{X\}, \{Y\}, \{Z, W\} \) and \( \{Z, W\} \) of lengths \( n, n, 2n - 1, 2n - 1 \) have zero non-periodic auto-correlation function, in \( \text{CS}(n; 2n - 1; 6n - 2) \).
<table>
<thead>
<tr>
<th>Lengths</th>
<th>Sums of Squares</th>
<th>$X, Y, Z, W$ of lengths $n, n, n, n-1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>23,12</td>
<td>$4^2 + 2^2 + 5^2 + 5^2$</td>
<td>$X = (- + + + + + - - + +)$, $Y = (- + + + - - + + +)$, $Z = (- - - + + + + + +)$, $W = (+ + + + - + - + -)$</td>
</tr>
<tr>
<td>27,14</td>
<td>$4^3 + 4^2 + 7^2 + 1^2$</td>
<td>$X = (- - - - + + + + + +)$, $Y = (- - - - + + + + + +)$, $Z = (- - - - + + + + + +)$, $W = (+ + + + - + - + -)$</td>
</tr>
<tr>
<td>27,14</td>
<td>$4^3 + 4^2 + 5^2 + 5^2$</td>
<td>$X = (- - - - - + + + + + +)$, $Y = (- - - - - + + + + + +)$, $Z = (- - - - - + + + + + +)$, $W = (+ + + + - + - + -)$</td>
</tr>
<tr>
<td>62,62</td>
<td>$6^2 + 6^2 + 3^2 + 1^2$</td>
<td>$X = (- + - - - - - - -)$, $Y = (+ - + + - - - -)$, $Z = (- + + + + + - -)$, $W = (+ - + + - - - -)$</td>
</tr>
</tbody>
</table>

These can be obtained by changing the signs of the odd elements of each sequence in $2t = 52 = 4^2 + 4^2 + 4^2 + 7^2$.

<table>
<thead>
<tr>
<th>Lengths</th>
<th>Sums of Squares</th>
<th>$X, Y, Z, W$ of lengths $n, n, n, n-1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>31,16</td>
<td>$4^4 + 2^2 + 7^2 + 5^2$</td>
<td>$X = (+ + - + + + - + + - +)$, $Y = (+ + - + + + - + + - +)$, $Z = (+ + + + + - + + - +)$, $W = (+ + + + + - + + - +)$</td>
</tr>
<tr>
<td>31,16</td>
<td>$0^2 + 6^2 + 7^2 + 3^2$</td>
<td>$X = (- - - - + + + + + +)$, $Y = (- - - - + + + + + +)$, $Z = (- - - - + + + + + +)$, $W = (- - - - + + + + + +)$</td>
</tr>
</tbody>
</table>

These can be obtained by changing the signs of the odd elements of each sequence in $2t = 34 = 2^4 + 2^2 + 5^2 + 7^2$.

<table>
<thead>
<tr>
<th>Lengths</th>
<th>Sums of Squares</th>
<th>$X, Y, Z, W$ of lengths $n, n, n, n-1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>35,18</td>
<td>$10^2 + 2^2 + 4^2 + 1^2$</td>
<td>$X = (+ + + + - + + + + - + + + +)$, $Y = (+ + + + - + + + + - + + + +)$, $Z = (+ + + + - + + + + - + + + +)$, $W = (+ + + + - + + + + - + + + +)$</td>
</tr>
<tr>
<td>35,18</td>
<td>$4^3 + 8^2 + 5^2 + 1^2$</td>
<td>$X = (+ + - - + + - + - + + +)$, $Y = (+ + - - + + - + - + + +)$, $Z = (+ + - - + + - + - + + +)$, $W = (+ + - - + + - + - + + +)$</td>
</tr>
</tbody>
</table>

These can be obtained by changing the signs of the odd elements of each sequence in $2t = 106 = 2^2 + 4^2 + 5^2 + 6^2$.

Table 1 (continued): $\{X\}, \{Y\}, \{Z, W\}$ and $\{Z, W\}$ of lengths $n, n, 2n-1$. $2n-1$ have zero non-periodic autocorrelation function, ie $CS(n, 2n-1; 6n-2)$.

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<table>
<thead>
<tr>
<th>Lengths</th>
<th>Sums of Squares</th>
<th>$X, Y, Z, W$ of lengths $n, n, n, n - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>39,20</td>
<td>$8^2 + 6^2 + 3^2 + 3^2$</td>
<td>$X = (+ + + - + + + + + + + - + - + +)$, $Y = (+ + - + + - + + + + + - +)$, $Z = (+ + - + + + + + + - + - +)$, $W = (+ + + + - + + + + + + + +)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X = (+ + + - + + + + + + + - + - + +)$, $Y = (+ + - + + - + + + + + - +)$, $Z = (+ + - + + + + + + - + - +)$, $W = (+ + + + - + + + + + + + +)$</td>
</tr>
<tr>
<td>43,22</td>
<td>$6^2 + 9^2 + 1^2$</td>
<td>$X = (+ + + - + + + + + + + - + - + +)$, $Y = (+ + - + + - + + + + + - +)$, $Z = (+ + - + + + + + + - + - +)$, $W = (+ + + + - + + + + + + + +)$</td>
</tr>
<tr>
<td>47,24</td>
<td>$2^2 + 4^2 + 11^2 + 1^2$</td>
<td>$X = (+ + + - + + + + + + + - + - + +)$, $Y = (+ + - + + - + + + + + - +)$, $Z = (+ + - + + + + + + - + - +)$, $W = (+ + + + - + + + + + + + +)$</td>
</tr>
</tbody>
</table>

Table 1 (continued): \{X\}, \{Y\}, \{Z, W\} and \{Z, -W\} of lengths $n, n, 2n - 1$, $2n - 1$ have zero non-periodic autocorrelation function, in $CS(n, 2n - 1; 6n - 2)$.

## 2 Multiplication theorems

We now summarize some constructions using sequences. Not all these constructions are new but they are collected here to help construct our table.

The following lemma, which closely follows work of Turyn, is to be found in [37].

**Lemma 4** If there exist

(i) $2 - CS(m_1, w_1)$ and $2k - DCS(m_2, w_2)$ then exist $2k - CS(m_1, m_2, w_1, w_2)$,

(ii) $2 - DCS(m_1, w_1)$ and $2k - CS(m_2, w_2)$ then exist $2k - CS(m_1, m_2, w_1, w_2)$,

(iii) $2 - DCS(m_1, w_1)$ and $2k - DCS(m_2, w_2)$ then exist $2k - DCS(m_1, m_2, w_1, w_2)$.

In the special case with $\overline{4}$ sequences where $A_2, A_3$ have length $m_2$ and $A_2, A_3$ have length $m_3$ we obtain $SS(m_2, m_3, m_4, w_2)$.

**Example 4** Suppose $A_1, A_2, A_3, A_4$ are $\{\overline{1} \ 1 \ 0 \ 0\}, \{0 \ 0 \ 1 \ 1\}, \{1 \ 0 \ 1 \ 0\}$ then if $X, Y$ are $2-CS(m, w)$ then

\{X, Y, Y, Y\}, \{X, Y, X\}, \{Y, Y, Y\} are $4-CS(m, w)$.  

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Suppose $A_1, A_2, A_3, A_4$ are \{1 \ 1 \ 0 \ 0\}, \{0 \ 0 \ 1 \ 1\}, \{1 \ 0 \ 1 \ 0\}, \{0 \ 0 \ 0 \ 1\}$ then if $X, Y$ are 2-CS($m, w$) then

$$\{XXXYY, XXYYY, X0XYY, XXY0Y\}$$

are 4-CS($5m, 9w$). So 2-CS(3,5) gives 4-CS(12,35) and 4-CS(15,45), 2-CS(6,10) gives 4-CS(24,70) and 4-CS(30,90), and 2-CS(11,13) gives 4-CS(44,91) and 4-CS(55,107).

**Example 5** We give some useful examples of the last lemma

(i) $2 - \text{CS}(3,5)$ and $2 - \text{DCS}(6,5)$ give $2 - \text{CS}(18,25)$;

(ii) $2 - \text{CS}(3,5)$ and $2 - \text{DCS}(14,13)$ give $2 - \text{CS}(42,65)$;

(iii) $2 - \text{CS}(3,5)$ and $2 - \text{DCS}(g, g)$ give $2 - \text{CS}(3g, 5g)$, $g$ a Golay number;

(iv) $2 - \text{CS}(11,13)$ and $2 - \text{DCS}(6,5)$ give $2 - \text{CS}(66,65)$;

(v) $2 - \text{CS}(11,13)$ and $2 - \text{DCS}(14,13)$ give $2 - \text{CS}(154,169)$;

(vi) $2 - \text{CS}(11,13)$ and $2 - \text{DCS}(g, g)$ give $2 - \text{CS}(11g, 13g)$, $g$ a Golay number;

(vii) $4 - \text{CS}(3,11)$ and $2 - \text{DCS}(6,5)$ give $4 - \text{CS}(18,55)$;

(viii) $2 - \text{DCS}(6,5)$ and $2 - \text{DCS}(6,5)$ give $2 - \text{DCS}(30,25)$ and $2 - \text{DCS}(6^* 5^*)^* \geq 0$;

(ix) $2 - \text{DCS}(14,13)$ and $2 - \text{DCS}(14,13)$ give $2 - \text{DCS}(196,169)$ and $2 - \text{DCS}(14^* 13^*)^* \geq 0$;

(x) $2 - \text{DCS}(6,5)$, $2 - \text{DCS}(14,13)$ and $2 - \text{DCS}(g, g)$ give $2 - \text{DCS}(6^* 14^* g), 5^* 13^* g)^* \geq 0$, $g$ a Golay number. \(\square\)

**Lemma 5** If there exist

(i) $2k - \text{CS}(m_1, w_1) A_1, A_2, \ldots, A_{2k} \text{ and } 2 - \text{CS}(m_2, w_2), X, Y$, then there are $2k - \text{CS}(2m_1 m_2, w_1 w_2)$;

(ii) $2k - \text{CS}(m_1, w_1) A_1, A_2, \ldots, A_{2k} \text{ and } 2 - \text{DCS}(m_2, w_2), X, Y$, then there are $2k - \text{DCS}(2m_1 m_2, w_1 w_2)$;

(iii) $2k - \text{DCS}(m_1, w_1) A_1, A_2, \ldots, A_{2k} \text{ and } 2 - \text{CS}(m_2, w_2), X, Y$, then there are $2k - \text{DCS}(2m_1 m_2, w_1 w_2)$.

**Proof.** Consider, where $i = 1, 2, \ldots, 2k$,

$$\{A_{2i} \times X, A_{2i+1} \times Y\} \text{ and } \{A_{2i} \times Y, A_{2i+1} \times -X\}$$

or

$$\{A_{2i} \times X, A_{2i+1} \times Y\} \text{ and } \{A_{2i} \times Y^*, A_{2i+1} \times -X^*\}$$

\(\square\)

**Example 6** 2-CS(3,5) and 2-CS(11,13) give 2-CS(66,65). \(\square\)
Lemma 8 If $X, Y$ are $2 - CS(m, w)$ (or $2 - DCS(m, w)$) then
$$\{aX, 0\}, \{aY, 0\}, \{a, 0_{1+m+1}\}, \{b, 0_{1+m+1}\}$$
are $4 - CS(m+1, 2w)$ when $a = b = c = 1$ (or $4 - DCS(m+1, 2w+2)$) and when $a, b, c$ are commuting variables these four sequences can be used to generate circulant matrices to use in the Goethals-Seidel array to form $OD(4(m+1); 1, 1, w)$ for all $t \geq 0$.

Example 7 2-CS$(3, 5)$ and 2 DCS$(10, 10)$ give 2-CS$(30, 50)$, by lemma 4, which can be used to give an $OD(4(30+t); 1, 1, 50)$, and 2-CS$(18, 25)$ can be used to give an $OD(4(18+t); 1, 1, 25)$.

Lemma 9 If $X, Y, Z, W$ are $4 - CS(m, w)$ (or $4 - DCS(m, w)$) then
$$\{aX, aW, 0\}, \{aX, aY, a_1\}, \{aZ, aW, 0\}, \{aZ, aY, 0\}$$
are $4 - CS(2m+4, 3w)$ when $a = b = c = 1$ (or $4 - DCS(2m+4, 3w+2)$) and when $a, b$ are commuting variables these four sequences can be used to generate circulant matrices to use in the Goethals-Seidel array to form $OD(4(2m+4); 1, 1, 3w)$ for all $t \geq 0$.

Lemma 10 Suppose there are 2-CS$(m, w)$, $X, Y$. Let $a, b, c$ be commuting variables. Then $\{aX, a, aY, 0\}, \{aX, a, cY, 0\}, \{aX, b, aY, 0\}, \{aX, b, cY, 0\}, \{aY, a, cX, 0\}, \{aY, a, cX, 0\}$ are the $4$-CS$(4m+2, 2m+4)$ which can be used to form the first rows of four circulant matrices which then used in the Goethals-Seidel array to form an $OD(4(2m+2+t); 1, 1, 4w)$ for all $t \geq 0$.

Example 8 There exist 2-DCS$(5, 8)$ and 2-CS$(n, 2u)$ where $u = 2^{10}2^{10}2^{10}2^{10}2^{10}2^{10}$ and $u = 2^{10}2^{10}2^{10}2^{10}2^{10}2^{10}$ or $2u$. By the lemma $OD(4(2u+t+1); 1, 1, 4u)$ exist for all $t \geq 0$. Hence we have
$$OD(4(17+t); 1, 1, 64), OD(4(14+t); 1, 1, 40), OD(4(23+t); 1, 1, 52)$$
$$OD(4(29+t); 1, 1, 104), OD(4(21+t); 1, 1, 80).$$

Giving the entries for weights $65$ and $81$ in the table. There exist $248(14, 13)$ and hence $OD(4(21+t); 1, 1, 52)$. Hence from [17, p406] Table II there are 4-sequences with zero periodic autocorrelation function which give useful results.

Lemma 9 $OD(4n; 1, 1, 50)$ exist for all $n \geq 13$. $OD(4n; 1, 52)$ and $OD(4n; 1, 1, 58)$ exist for all $n \geq 15$.

3 Some ad hoc multiplication lemmas

Lemma 10 Suppose $A, B, C$ are NS$(M)$ of weight $w$ then there are SS$(3n, 2w)$. If $H \pm C$ are $(0, 1, -1)$ sequences then there are TW$(3n, 3w)$ and $4$-CS$(2n, 3w)$.
Proof. We use the sequences
\[ \{ A, B, C \}, \{ A, -C \}, \{ A, 0_A \}, \{ A, -B \}, \{ A, B^* + C^* \}, \{ 0_{2m}, A, -B^* \}, \{ 0_{2m}, C, -A^* \}, \{ 0_{2m}, B \}, \{ A, -B \}, \{ A, -C \}, \{ B, -C \} \]
and
\[ \{ A, B + C \}, \{ A, -B \}, \{ A, -C \}, \{ B, -C \} \]
respectively. □

Lemma 11. Suppose \( A, B, C \) and \( D \) are 4 - CS(\( n, w \)) then there are
(i) \( 4 - CS(4m, 3w) \)
(ii) \( 4 - CS(6m, 5w) \)
(iii) \( 4 - DC(\{ n, w \}) \) give \( 4 - DC(6m, 5w) \).

Proof. We use the sequences
\[ \{ A, B, C \}, \{ A, -B, 0_{A}, D \}, \{ A, 0_{A}, -C, -D \}, \{ B, -C, D \} \]
and
\[ \{ A, A, -A, C, 0_{A}, C \}, \{ B, B, -B, 0_{A}, D \}, \{ A, 0_{A}, A, C, -C, -C \}, \{ B, 0_{A}, B, D, -D, -D \} \]
for (ii) and (iii) respectively. □

Example 9. The following 4 CS(3, 11) \( \{ \{ 1 \} \}, \{ \{ 1 \} \}, \{ \{ 1 \} \} \) have length
3 and total weight 11 so there are 4-CS(18, 55) of length 18 and total weight 55.
Using sequences of lengths 5 and weights 17 and 19 gives the sequences of
lengths 30 and weights 85 and 95 respectively.
The sequence of lengths 5, 6, 7, 8, 9 and 9 and weights 19, 21, 23, 25, 29,
and 33 give the results for weights 57, 63, 69, 75, 87, 93 and 99 and lengths
20, 24, 28, 32, 36 and 36 respectively. □

We state a theorem of Yang in the language used in [27] at the same time
deleting the terms from the beginnings or the ends of the constructed sequences:

4 Multiplication by \( y \) a Yang multiplier

Lemma 12. Let \( A, B, C, D \) be \( SS(p, n, m) \) and \( X, Y, G, H \) of lengths \( m, n, \)
\( 2m, 3n \) respectively be obtained from \( NN(4m + 1) \), then the following sequences
\( Q, R, S, T \) are TW sequences, \( TW(p + n) \), \( y = 4m + 1 \), where \( X = \{ x_1, x_2, \ldots, x_{m-1}, x_m \} \), \( Y = \{ y_1, y_2, \ldots, y_{m-1}, y_m \} \), \( G = \{ g_1, g_2, \ldots, g_{2m-1}, g_{2m} \} \),
and \( H = \{ h_1, h_2, \ldots, h_{2m-1}, h_{2m} \} \).

\[ Q = \{ Ag_{2m} - Bh_1, -Cg_1; Ag_{2m-1} - Bh_2, -D''x_1; \ldots; Ag_2 - Bh_{2m-1}, -Cg_2; \}
\]
\[ Ag_1 - Bh_{2m}, -D''x_0; 0', \ldots; 0' ; 0', 0; \cdots; 0', 0', 0 \} \]

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\[ R = \{ Bg_1 + Ab_{2m-1} - Dg_1; Bg_2 + Ah_{2m-1}; C^* z_1; \ldots; Bg_{2m-1} + Ah_1 - Dg_1; \}
\]
\[ Bg_{2m} + Ah_1, C^* z_m; 0', 0', 0', \ldots; 0', 0', 0', 0', 0', 0' \}
\]
\[ S = \{ 0'; 0'; 0'; \ldots; 0'; 0'; 0; -B; 0; -Bz_m, C^* g_{2m} + D^* h_{2m}; A^* g_1; \}
\[ C^* g_{2m-1} + D^* h_{2m-1}; \ldots; -Bz_1, C^* g_2 + D^* h_2; A^* y_m, C^* y_1 + D^* h_1 \}
\]
\[ T = \{ 0'; 0'; 0'; \ldots; 0'; 0'; 0'; 0; A; 0; Az_m, D^* y_1 - C^* h_1; B^* y_1, D^* y_2 - C^* h_2; \ldots; \}
\[ \ldots; Az_1, D^* y_{2m-1} - C^* h_{2m-1}; B^* y_m, D^* y_{2m} - C^* h_{2m} \}
\]
where \( 0', 0 \) are sequences of zeros of length \( p \) and \( n \) respectively. If \( A, B, C, D \) are \( SS(p, n; 2p + 2n) \) then \( Q, R, S, T \) are \( T \)-sequences.

\[ Q_1 = \{ Ag_{2m} - Bh_1, C^* y_1; Ag_{2m-1} - Bh_2, D^* z_1; \ldots; Ag_{1} - Bh_{2m-1} - C^* y_m; \}
\]
\[ Ag_1 - Bh_{2m} - D^* z_m; 0', -D^* \}
\]
\[ R_1 = \{ Bg_1 + Ah_{2m} - Dg_1; Bg_2 + Ah_{2m-1}; C^* z_1; \ldots; Bg_{2m-1} + Ah_1 - Dg_1; \}
\]
\[ Bg_{2m} + Ah_1, C^* z_m; 0', C^* \}
\]
\[ S_1 = \{ -B; 0; -Bz_m, C^* g_{2m} + D^* h_{2m}; A^* y_1, C^* g_{2m-1} + D^* h_{2m-1}; \ldots; -Bz_1, \}
\[ C^* g_2 + D^* h_2; A^* y_m, C^* y_1 + D^* h_1 \}
\]
\[ T_1 = \{ A; 0; Az_m, D^* y_1 - C^* h_1; B^* y_1, D^* y_2 - C^* h_2; \ldots; Az_1, D^* y_{2m-1} - C^* h_{2m-1}; \}
\[ D^* y_{2m}, D^* y_{2m} - C^* h_{2m} \}
\]
of lengths \((y + 1)(p + n)/2\) and total weight \( yw \) in \( SS((y + 1)(p + n)/2, yw) \).

Now since \( Q, R, S, T \) are \( TW(Y(m + n); yw) \)

\[ \{ Q, R^*, \{ S, T^* \}, \{ T, S^* \} \}
\]

are \( TW(Y(p + n); yw) \). These \( TW \)-sequences can be used to form circulant matrices in the Goethals-Seidel array which give \( OD(Y; yw, yw, yw, yw) \) for \( s \geq y(p + n) \) and \( OD(Y; yw, yw, yw, yw) \) for \( t \geq 2y(p + n) \).

Let \( a \) and \( b \) be commuting variables. Then \( aQ_1 + bR_1, bQ_1 - aR_1, aS_1 + bT_1, aS_1 - bT_1 \) can be used to form circulant matrices used in the Goethals-Seidel array which give \( OD(4w, yw, yw) \) for \( u \geq (y + 1)(p + n)/2 \).

Lemma 13 If there exist \( SS(p, n; w) \) and \( NN(y) \) then there exist \( SS((y + 1)(p + n)/2, yw) \).

Example 10 Let \( E = \{ 1, 1 \}, P = \{ 1 \}, G = \{ 1, \} \), \( H = \{ 0, 0 \} \).

Then \( z_1 = 1, y_1 = 1, g_1 = 1, g_2 = -1, h_1 = 0 \) and \( h_2 = 0 \) giving

\[ Q_1 = \{ -A, -C; A, -D^*; 0, -D^* \}
\]
\[ R_1 = \{ B, -D^*; -B, C^*; 0, C^* \}
\]
\[ S_1 = \{ -B, 0; -B, C^*; A^*, C^* \}
\]
\[ T_1 = \{ A, 0; A, D^*; B^*; -D^* \}
\]

show that \( 4-CS(n; w) \) and \( NN(5) \) give \( 4-CS(6m; 5w) \) and \( 4-CS(m, n; w) \) and \( NN(5) \) give \( 4-CS(3(m + n; 5w)) \).
Example 11 4-CS(5;17) and 4-CS(5;19) can be used with NN(3) to obtain 4-
CS(20;85) and 4-CS(30;95) respectively.

Lemma 14 If there exist 4-CS(n;w) and NN(y) then there exist 4-CS((y +
1)n;yw). □

Example 12 4-CS(3,7) gives 4-CS(42,91).

5 Multiplication by 3

Lemma 15 Let A, B, C, D be SS(m,n;w). Then consider the following sets of
four sequences, where 00 is the sequence of m + n zeros and 0 is the sequence of
n zeros,

\[ X = \{AC; 00; B*0 \} \]
\[ Y = \{BD; 00; A*0 \} \]
\[ Z = \{00; AC; 0D* \} \]
\[ W = \{00; B*D; 0C* \} \]

are SS(3(m + n),3(m + n);3w). Note X, Y, Z, W are TW(3(m + n);3w) and
\[ \{X, Y^*, \{Y, X^*, \{Z, W^*, \{W, Z^* \} \} \}

are TW((6(m+n));6w). These TW-sequences can be used to form circulant matrices
used in the Goethals-Seidel array which give OD(4;3w,3w,3w,3w) for s ≥ 3(m + n) and OD(4;6w,6w,6w,6w) for t ≥ 6(m + n). Further we note that
zeros can be eliminated giving

\[ P = \{AC; 00; B^* \} \]
\[ Q = \{BD; 00; A^* \} \]
\[ R = \{AC; 0D^* \} \]
\[ S = \{BD; 0C^* \} \]

of lengths 3m + 2n, 3m + 2n, 2m + 2n, 2m + 2n and total weight 3w in 4-
CS(3m + 2n;3w). There are also SS(3m + 2n,2m + 2n,3w) and can be used recursively in the construction.

Let a and b be commuting variables. Then aP + bQ, bP - aQ, aR + bS, 
aH - bS can be used to form circulant matrices used in the Goethals-Seidel array
which give OD(4;3w,3w) for s ≥ 3m + 2n.

Example 13 Let A = \{10\}, B = \{01\}, C = \{1\}, D = \{0\}.

\[ X = \{101;009;100 \} \]
\[ Y = \{010;000;0-0 \} \]
\[ Z = \{000;10-;000 \} \]
\[ W = \{000;010;00- \} \]
are $SS(9,9;9)$ and are also $TW(9;9)$ and give $OD(4t;9,9,9,9)$, for $4t \geq 36$.

Further

\[ \{X,Y^{*}\}, \{Y^{*},X^{*}\}, \{Z,W^{*}\}, \{W^{*},Z^{*}\} \]

are $TW(18;18)$ and give $OD(4s;18,18,18,18)$, for $4s \geq 72$.

\[ P = \{101;000;10\} \]

\[ Q = \{010;000;0--\} \]

\[ R = \{10--;000\} \]

\[ S = \{010,000\} \]

are 4-complementary sequences of 8, 8, 6 and 6 and total weight 9 and zero autocorrelation function. Furthermore they are $SS(8,8,9)$.

Using the construction recursively we get $TW(42;27)$ and $SS(36,18;27)$.

We note that in this case the sequences can be shortened further to give 4-complementary sequences $P_1 = \{1010001\}$, $Q_1 = \{100010--\}$, $R_1 = \{10--;000\}$, $S_1 = \{000--;00\}$ of lengths 7, 7, 3, 5 and total weight 9.

\[ \square \]

Lemma 10 Suppose $A$, $B$, $C$, and $D$ are 4-CS(m,n;w) of lengths $m$, $n$, $n$, $n$ where $m \geq n$ then

\[ \{ A_m, A, C, 0_m, 0_n, B^{*}\}, \{ A_m, 0_m, 0_n, -B^{*}, -B^{*}, 0_n\} \]

and

\[ \{ A, -C, 0_m, D^{*}, 0_n, 0_n\}, \{ 0_m, 0_n, C, 0_n, D^{*}, -B^{*}\} \]

are $SS(4m+2n,3m+3n;3w)$, or, prefixing the last two sequences by $0_{2n+2m}$, $TW(7m+5n;3w)$.

6 Multiplication by 7

Lemma 17 Let $A$, $B$, $C$, $D$ be $SS(m,n;w)$. Then consider the following sets of four sequences, where $\emptyset$ is the sequence of $m+n$ zeros and 0 is the sequence of $n$ zeros,

\[ X = \{AC;00;AD;00;AC;00;B^{*}0\} \]

\[ Y = \{BD;00;BC;00;BD;00;A^{*}0\} \]

\[ Z = \{00;AC;00;BC;00;AC;00A^{*}\} \]

\[ W = \{00;BD;00;AD;00;BD;00C^{*}\} \]

all of length $7(m+n)$ and total weight $7w$, or $SS(7m+n,7m+n;7w)$. Now $X$, $Y$, $Z$, $W$ are $TW(7(m+n);7w)$ and

\[ \{X,Y^{*}\}, \{Y^{*},X^{*}\}, \{Z,W^{*}\}, \{W^{*},Z^{*}\} \]

are $TW(14(m+n);14w)$. These $TW$-sequences can be used to form circulant matrices used in the Goethals-Seidel array which give $OD(4s;7w,7w,7w;7w)$ for $s \geq 7(m+n)$ and $OD(4s;14w,14w,14w;14w)$ for $s \geq 14(m+n)$.

Further we note that zeros can be eliminated giving

\[ P = \{AC;00;AD;00;AC;00;B^{*}\} \]

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\[ Q = \{ BD; 00; BC; 00; BD; 06; A^* \} \]
\[ R = \{ AC; 00; BC; 00; AC; 0D^* \} \]
\[ S = \{ BD; 00; AD; 06; BD; 0C^* \} \]
of lengths \(7m + 6n, \ 7m + 6n, \ 6m + 6n, \ 6m + 6n\) and total weight \(7m\). These are \(S3(7m + 6n, 6m + 6n, 7m)\) and can be used recursively in the construction.

Let \(a\) and \(b\) be commuting variables. Then \(aP + bQ, bP - aQ, aR + bS, aR - bS\) can be used to form circulant matrices used in the Grotthuss-Scott array which give \(OD(4u; 7w, 7w)\) for \(u \geq 7m + 6n\).

If \(A, B, C, D\) are \(4 - CS(n; w)\) then \(P, Q, R, S\) are \(4 - CS(13n; 7w)\).

**Example 14** Let \(A = \{11 - 0\}, \ B = \{0001\}, \ C = \{011\}, \ D = \{010\}\), then
\[ X = \{1 - 10010; 0; 11 - 0010; 0; 11 - 0101; 0; 11 - 000000\} \]
\[ Y = \{0000 - 010; 0; 00001 - 0; 0; 00001010; 0; 0 - 110000\} \]
\[ Z = \{0; 11 - 0 - 0; 0; 0000 - 0; 0; 11 - 0101; 0; 00000 - 0\} \]
\[ W = \{0; 0000 - 0; 0; 11 - 00 - 0; 0; 00001010; 0; 00000101\} \]
are \(TW(40; 40)\) and give \(OD(44; 49, 49, 49)\), for \(d \geq 196\). Further
\[ \{X, Y^*\}, \{X, Y^*, Z, W^*\}, \{W, Z^*\} \]
are \(TW(98; 98)\) and give \(OD(44; 98, 98, 98, 98)\), for \(4s \geq 392\).

We also note that
\[ P = \{ - 10101; 0, 7; 11 - 0010; 0, 7; 11 - 0101; 0, 7; - 0000\} \]
\[ Q = \{0000 - 010; 0; 00001 - 0; 0; 00001010; 0; 0 - 110000\} \]
\[ R = \{11 - 0 - 0; 0; 0000 - 0; 0; 11 - 0101; 0; 00000 - 0\} \]
\[ S = \{000010 - 0; 0; 11 - 00 - 0; 0; 00001010; 0; 0000101\} \]
give sequences of length \(46\) and weight \(49\) and hence \(OD(4u; 49, 49)\) for \(4u \geq 184\). Careful observation shows that \(P, Q, R, S\) can all be further shortened by removing zero from both the beginning and the end of each sequence giving
\[ R_1 = \{ - 101010; 0, 7; 11 - 0010; 0, 7; 11 - 0101; 0, 7; - 0\} \]
\[ Q_2 = \{000010 - 0; 0; 00001010; 0; 0 - 110000\} \]
\[ R_2 = \{11 - 0 - 0; 0; 00001010; 0; 00000101\} \]
give sequences of length \(43\) and weight \(49\) and hence \(OD(4u; 49, 49)\) for \(4u \geq 172\). \(\square\)

**Corollary 2** All \(OD(4s; 98, 98, 98, 98)\) exist for \(4s \geq 392\). Hence all \(W(4s; 392)\) exist for \(4s \geq 392\).

**Example 15** Using the \(SS(3, 4; 7)\) \(\{101\}, \{010\}, \{11 - 0\}, \{0001\}\) and using the \(HS(4)\) we obtain \(SS(45, 49, 91)\). \(\square\)

**Example 18** \(4 - CS(3, 7)\) gives \(4 - CS(39, 49)\), \(4 - CS(3, 11)\) gives \(4 - CS(39, 77)\) and \(4 - CS(4, 13)\) gives \(4 - CS(52, 91)\). \(\square\)
Table 2: Sequences of length 13 with zero periodic autocorrelation function

<table>
<thead>
<tr>
<th>Length=13</th>
<th>Sequences with zero periodic autocorrelation function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,42</td>
<td>{−++−−−0a0++−−+},</td>
</tr>
<tr>
<td></td>
<td>{−−−−0+0+++−+},</td>
</tr>
<tr>
<td></td>
<td>{++−−+−0000+−},</td>
</tr>
<tr>
<td></td>
<td>{++++−−000++−}</td>
</tr>
</tbody>
</table>

Table 3: Sequences of length 15 with zero non-periodic autocorrelation function

<table>
<thead>
<tr>
<th>Length=15</th>
<th>Sequences with zero non-periodic autocorrelation function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,42</td>
<td>{0+0+-−−a−+−−0−0},</td>
</tr>
<tr>
<td></td>
<td>{++0+++0−0+00+},</td>
</tr>
<tr>
<td></td>
<td>{000++0++0−−−−+},</td>
</tr>
<tr>
<td></td>
<td>{−+−+−++0++−−−},</td>
</tr>
<tr>
<td>1,48</td>
<td>{−++−−−−0−a++−−−},</td>
</tr>
<tr>
<td></td>
<td>{++0−+−000+−++−0},</td>
</tr>
<tr>
<td></td>
<td>{−−++0+0+0+−−−−+0},</td>
</tr>
<tr>
<td></td>
<td>{++−−−−+0++−+−−++},</td>
</tr>
<tr>
<td>1,54</td>
<td>{−−−−−−−0++−−−−−−},</td>
</tr>
<tr>
<td></td>
<td>{−−−−++0++−−−+0},</td>
</tr>
<tr>
<td></td>
<td>{−++−−−−−0++−−−−0},</td>
</tr>
<tr>
<td></td>
<td>{−−−−++0++−−−−0},</td>
</tr>
<tr>
<td>55</td>
<td>{−++−−−−−0++−−−−0},</td>
</tr>
<tr>
<td></td>
<td>{−−−−−−++0++−−−−0},</td>
</tr>
<tr>
<td></td>
<td>{−++−−−−−0++−−−−0},</td>
</tr>
<tr>
<td></td>
<td>{−−−−++0++−−−−0},</td>
</tr>
<tr>
<td></td>
<td>{++−−−−−−−0++−−−−0},</td>
</tr>
<tr>
<td></td>
<td>{++−−−−−−−0++−−−−0},</td>
</tr>
<tr>
<td></td>
<td>{++−−−−−−−0++−−−−0},</td>
</tr>
<tr>
<td></td>
<td>{++−−−−−−−0++−−−−0},</td>
</tr>
<tr>
<td>56</td>
<td>{−−−−−−−0++−−−−0},</td>
</tr>
<tr>
<td></td>
<td>{−−−−−−−0++−−−−0},</td>
</tr>
<tr>
<td></td>
<td>{−−−−−−−0++−−−−0},</td>
</tr>
<tr>
<td></td>
<td>{−−−−−−−0++−−−−0},</td>
</tr>
<tr>
<td></td>
<td>{−−−−−−−0++−−−−0},</td>
</tr>
<tr>
<td>59</td>
<td>{−−−−−−−0−−−−−−−},</td>
</tr>
<tr>
<td></td>
<td>{−−−−−−−0−−−−−−−},</td>
</tr>
<tr>
<td></td>
<td>{−−−−−−−0−−−−−−−},</td>
</tr>
<tr>
<td></td>
<td>{−−−−−−−0−−−−−−−},</td>
</tr>
<tr>
<td></td>
<td>{−−−−−−−0−−−−−−−},</td>
</tr>
<tr>
<td></td>
<td>{−−−−−−−0−−−−−−−},</td>
</tr>
</tbody>
</table>

116
<table>
<thead>
<tr>
<th>Length=15</th>
<th>Sequences with zero periodic autocorrelation function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,48</td>
<td>(+ + 0 - - a + + 0 - -),</td>
</tr>
<tr>
<td></td>
<td>(0 + + + + 0 0 + + + + + 0),</td>
</tr>
<tr>
<td></td>
<td>(- + + + + 0 0 + + - - - + 0),</td>
</tr>
<tr>
<td></td>
<td>(- + + 0 + + 0 + + 0 - + -)</td>
</tr>
<tr>
<td>1,56</td>
<td>(a - - - - + - - - + + +),</td>
</tr>
<tr>
<td></td>
<td>(- + + + + + + + - - - + + 0),</td>
</tr>
<tr>
<td></td>
<td>(+ + + + + 0 - - - - + + +),</td>
</tr>
<tr>
<td></td>
<td>(+ + + + + 0 - - - - + + -)</td>
</tr>
</tbody>
</table>

Table 4: Sequences of length 15 with zero periodic autocorrelation function

<table>
<thead>
<tr>
<th>Length=17</th>
<th>Sequences with zero non-periodic autocorrelation function</th>
</tr>
</thead>
<tbody>
<tr>
<td>61</td>
<td>(- + + - + + + + 0 - + 0 - + -),</td>
</tr>
<tr>
<td></td>
<td>(+ - + 0 + + + + - - - + + -),</td>
</tr>
<tr>
<td></td>
<td>(- + + + + + - 0 - - + + + + + -),</td>
</tr>
<tr>
<td></td>
<td>(+ + 0 - 0 - 0 + + - + 0 - +)</td>
</tr>
<tr>
<td>62</td>
<td>(0 + + + + - + + + - - + + - +),</td>
</tr>
<tr>
<td></td>
<td>(0 - + + + + + + + - 0 + + -),</td>
</tr>
<tr>
<td></td>
<td>(+ + 0 - + + + + + + - 0 + + -),</td>
</tr>
<tr>
<td></td>
<td>(- - - 0 + + + + + - + + - +),</td>
</tr>
<tr>
<td></td>
<td>(0 + + + + - + + + + + + + -),</td>
</tr>
<tr>
<td>63</td>
<td>(+ + - + + + - 0 + + + + + -),</td>
</tr>
<tr>
<td></td>
<td>(0 - + + - 0 - - + + + + + + -),</td>
</tr>
<tr>
<td></td>
<td>(+ + + + + - + + 0 - + - 0)</td>
</tr>
</tbody>
</table>

Table 5: Sequences of length 17 with zero non-periodic autocorrelation function
<table>
<thead>
<tr>
<th>Length=17</th>
<th>Sequences with zero periodic autocorrelation function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,42</td>
<td>{ + 0 + 0 + - - a + + - - 0 + 0 }</td>
</tr>
<tr>
<td></td>
<td>{ + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 }</td>
</tr>
<tr>
<td></td>
<td>{ + 0 + 0 + 0 - - - + - 0 + 0 + 0 + 0 }</td>
</tr>
<tr>
<td>1,49</td>
<td>{ + 0 + a - 0 + 0 + 0 - 0 + 0 + 0 }</td>
</tr>
<tr>
<td></td>
<td>{ + 0 + + + - + 0 + 0 + 0 + 0 + 0 + 0 + 0 }</td>
</tr>
<tr>
<td></td>
<td>{ + 0 + + + - + 0 + 0 + 0 + 0 + 0 + 0 + 0 }</td>
</tr>
<tr>
<td>1,57</td>
<td>{ + 0 + + - - - a + 0 + 0 + 0 + 0 + 0 }</td>
</tr>
<tr>
<td></td>
<td>{ + 0 + - - - a + 0 + 0 + 0 + 0 + 0 + 0 }</td>
</tr>
<tr>
<td>1,61</td>
<td>{ + 0 + + - - - a + - - - 0 + 0 + 0 }</td>
</tr>
<tr>
<td></td>
<td>{ + 0 + - - - a + - - - 0 + 0 + 0 + 0 }</td>
</tr>
<tr>
<td>1,62</td>
<td>{ + 0 + + - - - a + - - - 0 + 0 + 0 + 0 }</td>
</tr>
<tr>
<td></td>
<td>{ + 0 + - - - a + - - - 0 + 0 + 0 + 0 + 0 }</td>
</tr>
<tr>
<td>1,64</td>
<td>{ + 0 + + - - - a + - - - 0 + 0 + 0 + 0 }</td>
</tr>
<tr>
<td></td>
<td>{ + 0 + + - - - a + - - - 0 + 0 + 0 + 0 + 0 }</td>
</tr>
<tr>
<td>1,67</td>
<td>{ + 0 + + - - - a + - - - 0 + 0 + 0 + 0 }</td>
</tr>
<tr>
<td></td>
<td>{ + 0 + + - - - a + - - - 0 + 0 + 0 + 0 + 0 }</td>
</tr>
</tbody>
</table>

Table 6: Sequences of length 17 with zero periodic autocorrelation function

<table>
<thead>
<tr>
<th>Length=2t+15, t \geq 0</th>
<th>Sequences with zero non-periodic autocorrelation function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,56</td>
<td>{ + 0 + + + + + + + + + + + + + + + + + + }</td>
</tr>
<tr>
<td></td>
<td>{ + 0 + + + + + + + + + + + + + + + + + + }</td>
</tr>
<tr>
<td></td>
<td>{ + 0 + + + + + + + + + + + + + + + + + + }</td>
</tr>
</tbody>
</table>

Table 7: Sequences of length 2t+15, t \geq 0 with zero non-periodic autocorrelation function
<table>
<thead>
<tr>
<th>Length</th>
<th>Weight</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>7</td>
<td>[17, Table H.1 and H.2]</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>[17, Table H.1 and H.2]</td>
</tr>
<tr>
<td>[w/4]</td>
<td>w ≤ 20</td>
<td>w ≠ 7, 15 [17, Table H.1 and H.2]</td>
</tr>
<tr>
<td>6</td>
<td>21</td>
<td>[17]</td>
</tr>
<tr>
<td>6</td>
<td>22</td>
<td>base sequences</td>
</tr>
<tr>
<td>7</td>
<td>23</td>
<td>[17]</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>double sequences of length 3 [17, p401]</td>
</tr>
<tr>
<td>7</td>
<td>25</td>
<td>[17]</td>
</tr>
<tr>
<td>7</td>
<td>26</td>
<td>[17]</td>
</tr>
<tr>
<td>7</td>
<td>27</td>
<td>[17]</td>
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<tr>
<td>7</td>
<td>28</td>
<td>[17]</td>
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<tr>
<td>8</td>
<td>29</td>
<td>[17]</td>
</tr>
<tr>
<td>8</td>
<td>30</td>
<td>[17]</td>
</tr>
<tr>
<td>9</td>
<td>31</td>
<td>[17]</td>
</tr>
<tr>
<td>8</td>
<td>32</td>
<td>[17]</td>
</tr>
<tr>
<td>9</td>
<td>33</td>
<td>[17]</td>
</tr>
<tr>
<td>9</td>
<td>34</td>
<td>[17]</td>
</tr>
<tr>
<td>9</td>
<td>35</td>
<td>[17]</td>
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<tr>
<td>9</td>
<td>36</td>
<td>[17]</td>
</tr>
<tr>
<td>10</td>
<td>37</td>
<td>[17]</td>
</tr>
<tr>
<td>10</td>
<td>38</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>10</td>
<td>39</td>
<td>[17]</td>
</tr>
<tr>
<td>10</td>
<td>40</td>
<td>Golay sequences [17]</td>
</tr>
<tr>
<td>13</td>
<td>41</td>
<td>[17]</td>
</tr>
<tr>
<td>11</td>
<td>42</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>11</td>
<td>43</td>
<td>[17]</td>
</tr>
<tr>
<td>11</td>
<td>44</td>
<td>[17]</td>
</tr>
<tr>
<td>12</td>
<td>45</td>
<td>[17]</td>
</tr>
<tr>
<td>12</td>
<td>46</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>13</td>
<td>47</td>
<td>[17]</td>
</tr>
<tr>
<td>12</td>
<td>48</td>
<td>double sequences of length 6 weight 24</td>
</tr>
<tr>
<td>15</td>
<td>49</td>
<td>Table 3</td>
</tr>
<tr>
<td>13</td>
<td>50</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>13</td>
<td>51</td>
<td>[17]</td>
</tr>
<tr>
<td>13</td>
<td>52</td>
<td>[17]</td>
</tr>
<tr>
<td>15</td>
<td>53</td>
<td>[29]</td>
</tr>
<tr>
<td>13</td>
<td>54</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>15</td>
<td>55</td>
<td>Table 3</td>
</tr>
<tr>
<td>14</td>
<td>56</td>
<td>double sequences of length 7 weight 28</td>
</tr>
<tr>
<td>15</td>
<td>57</td>
<td>[29]</td>
</tr>
<tr>
<td>15</td>
<td>58</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>15</td>
<td>59</td>
<td>Table 3</td>
</tr>
<tr>
<td>15</td>
<td>60</td>
<td>base sequences of lengths 8, 8, 7, 7: remark 1</td>
</tr>
</tbody>
</table>

Table 8: The smallest length for which there are four sequences with zero non-periodic autocorrelation function.
<table>
<thead>
<tr>
<th>Length</th>
<th>Weight</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>61</td>
<td>Table 5</td>
</tr>
<tr>
<td>16</td>
<td>62</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>17</td>
<td>63</td>
<td>Table 5</td>
</tr>
<tr>
<td>16</td>
<td>64</td>
<td>[17]</td>
</tr>
<tr>
<td>17</td>
<td>65</td>
<td>Golay: example 8</td>
</tr>
<tr>
<td>17</td>
<td>66</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>18</td>
<td>67</td>
<td>[17]</td>
</tr>
<tr>
<td>18</td>
<td>68</td>
<td>[17]</td>
</tr>
<tr>
<td>18</td>
<td>69</td>
<td>[17]</td>
</tr>
<tr>
<td>18</td>
<td>70</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>24</td>
<td>71</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>18</td>
<td>72</td>
<td>double sequences of length 9 weight 36</td>
</tr>
<tr>
<td>19</td>
<td>73</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>74</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>25</td>
<td>75</td>
<td>example 3</td>
</tr>
<tr>
<td>20</td>
<td>76</td>
<td>double sequences of length 10 weight 38</td>
</tr>
<tr>
<td>39</td>
<td>77</td>
<td>example 16</td>
</tr>
<tr>
<td>20</td>
<td>78</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>79</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>80</td>
<td>Golay sequences: [17]</td>
</tr>
<tr>
<td>21</td>
<td>81</td>
<td>Golay: example 8</td>
</tr>
<tr>
<td>21</td>
<td>82</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>83</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>84</td>
<td>base sequences of lengths 11,11,10,10: remark 1</td>
</tr>
<tr>
<td>30</td>
<td>85</td>
<td>example 9</td>
</tr>
<tr>
<td>22</td>
<td>86</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>29</td>
<td>87</td>
<td>example 3</td>
</tr>
<tr>
<td>22</td>
<td>88</td>
<td>double sequences of length 11 weight 44</td>
</tr>
<tr>
<td>89</td>
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<td></td>
</tr>
<tr>
<td>23</td>
<td>90</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>44</td>
<td>91</td>
<td>example 4</td>
</tr>
<tr>
<td>23</td>
<td>92</td>
<td>from base sequences of lengths 12,12,11,11: remark 1</td>
</tr>
<tr>
<td>31</td>
<td>93</td>
<td>example 3</td>
</tr>
<tr>
<td>24</td>
<td>94</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>30</td>
<td>95</td>
<td>example 9</td>
</tr>
<tr>
<td>24</td>
<td>96</td>
<td>double sequences of length 12 weight 48</td>
</tr>
<tr>
<td>97</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>98</td>
<td>base sequences: remark 1</td>
</tr>
<tr>
<td>33</td>
<td>99</td>
<td>example 3</td>
</tr>
<tr>
<td>25</td>
<td>100</td>
<td>base sequences of lengths 13, 13, 12, 12: remark 1</td>
</tr>
</tbody>
</table>

Table 8 (continued): The smallest length for which there are four sequences with zero non-periodic autocorrelation function.
7 Numerical consequences

We use the tables of Appendix H of Geramita and Seberry [17] with non-periodic autocorrelation function zero and note that sequences with periodic autocorrelation function zero exist for $W(4t, 4t - 1), W(4t, 4t)$ for all $t \in \{1, \ldots, 31\}$ [27]. We extend Theorem 4.149 of Geramita and Seberry [17] using the results of [29] and those given in Tables 2 and 3.

**Theorem 2** There exists an orthogonal design $OD(4n; 1, k)$ when

(i) for $n \geq t, t = 3, 5, 7, 9$ with $k \in \{x : x \leq 4t - 1, x = a^2 + b^2 + c^2\}$; 
(ii) for $n \geq 11$, with $k \in \{x : x \leq 43, x = a^2 + b^2 + c^2, x \neq 42\}$; 
(iii) for $n \geq 13$, with $k \in \{x : x \leq 51, x = a^2 + b^2 + c^2, x \neq 46, 49\}$; 
(iv) for $n \geq 15$, with $k \in \{x : x \leq 59, x = a^2 + b^2 + c^2, x \neq 46, 49, 57\}$; 
(v) for $n \geq 17$, with $k \in \{x : x \leq 67, x = a^2 + b^2 + c^2, x \neq 46, 49, 57, 60, 61, 62, 66, 67\}$.

All are constructed by using four circulant matrices in the Goethals-Seidel array.

**Proof.** The sequences for $OD(4n; 1, 34)$, $n \geq 13$ are given in [29]. The $OD(36; 1, 34)$ is given in [17, Lemma 8.31] and the $OD(42; 1, 34)$ is given in [17, Lemma 8.36]. This completes the case for $t = 9$.

The sequences for $OD(4n; 1, 37)$, $n \geq 13$ are given in [29]. The $OD(44; 1, 37)$ is given in [17, Lemma 8.36]. This gives case (ii).

The sequences for $OD(52; 1, 42)$, are given in Table 2, while the sequences for $OD(4n; 1, 42)$, $n \geq 13$ are given in Table 3. The sequences for $OD(4n; 1, 45)$, $n \geq 13$, $OD(4n; 1, 53)$ and $OD(4n; 1, 56)$, $n \geq 15$ are given in [29]. The sequences for $OD(4n; 1, 48)$ and $OD(4n; 1, 54)$, $n \geq 15$, are given in Table 3. Hence noting $OD(52; 1, 48)$ is given in [29] we have cases (ii), (iii) and (iv).

Now there are Golay sequences of length 8 and total weight 16 so by [17, Lemma 4.118] we have $OD(4n; 1, 1, 64)$, $n \geq 17$.

$OD(4n; 1, 66)$ and $OD(4n; 1, 68)$, $n \geq 18$ are given in [17, Table II.2].

**Theorem 3** There exists a $W(4n; k)$ when

(i) for $n \geq 4, t = 3, 5, 7, 9, 11$ with $k \in \{x : x \leq 4t\}$; 
(ii) for $n \geq 13$, with $k = 1, \ldots, 52$; 
(iii) for $n \geq 15$, with $k = 1, \ldots, 60$; 
(iv) for $n \geq 17$, with $k = 1, \ldots, 68$; 
(v) for $n \geq 19$, with $k = 1, \ldots, 70, 72, 74, 76$.

All are constructed by using four circulant matrices in the Goethals-Seidel array.

**Proof.** Case (i) follows from the previous theorem by setting variables equal to each other or to zero.

The previous theorem gives us all values for case (ii) except 47 which may be found in [17, Table II.2].

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Given the results in case (ii) we need to consider \( k = 53, 54, 55, 56, 57, 58, 59 \) and 60. The result for 58, 59 and 60 can be found in [17, Table H.4], the result for 53, 54, 56, 57 can be found in [29]. We give the result for 55 in Table 3.

For case (iv) we consider 61, 62, 63, 64, 65, 66, 67 and 68. The \( OD(4m; 1, 1, 64) \), \( n \geq 17 \), mentioned in the proof of the previous Theorem gives 64, 65 and 66. There is an \( OD(68; 1, 67) \) (from good matrices [37]) and \( OD(4n; 1, 2, 66) \), \( n \geq 18 \) are given in [17, Table H.2] giving the result for 67 and 68. The sequences for 61, 62 and 63 are given in Table 5 for \( n \geq 17 \). This completes case (iv) and gives 69 for \( n \geq 18 \).

We note that if \( \{A\}, \{B\}, \{C\} \) and \( \{D\} \) are sequences of lengths \( m+1, m, m+1 \) and \( m \) and weight \( w \) then \( \{A, B\}, \{A, -B\}, \{C, D\} \) and \( \{C, -D\} \) are sequences of length \( 2m+1 \) and weight \( 2w \). Now [17, Table H.2] gives us sequences of this type for \( m = 9 \) with weights \( w = 35, 36, 37 \) and 38 giving sequences of length \( \geq 19 \) for 71, 72, 74 and 76.

**Lemma 18** There exists a \( W(52, k) \) for \( k \in \{x : 0 \leq x \leq 52\} \). All may be constructed from four circulant matrices using the Goethals-Seidel array.

**Proof.** Geanina and Seberry [17] give a \( W(52, k) \) for all \( k \neq 46 \). Table 8 shows that \( W(44, 46) \) exist for all \( t \geq 12 \) and may be constructed using sequences to give first rows of circulant matrices to use in the Goethals-Seidel array giving the result.

**Lemma 19** There exists a \( W(60, k) \) for \( k \in \{x : 0 \leq x \leq 60\} \). All are constructed from four circulant matrices in the Goethals-Seidel array.

**Proof.** Use Theorem 3.

**Lemma 20** There exists a \( W(68, k) \) for \( k \in \{x : 0 \leq x \leq 68\} \). All are constructed from four circulant matrices in the Goethals-Seidel array.

**Proof.** Use Theorem 3.

**Lemma 21** There exists a \( W(76, k) \) for all \( k \) except possibly 71, 73, which are undecided. All are constructed by using four circulant matrices in the Goethals-Seidel array.

**Proof.** Theorem 3 gives all values except 71, 73 and 75. The result for \( W(76, 75) \) is obtained from [37].

**Lemma 22** There exists a \( W(84, k) \) for all \( k \) except possibly 71, 79, which are undecided. All but \( W(84, 77) \) are constructed by using four circulant matrices in the Goethals-Seidel array.

**Proof.** This result appears in [35] except for the circulant property.

The Theorem 3 gives all values except 71, 73, 75, 77, 78, 79, 80, 81, 82, 83, 84. From [17, p.330] proof of Lemma 8.20, we see \( W(42, k) \) constructed from two circulants exist for \( k \in \{0, 1, 2, 4, 5, 8, 16, 13, 16, 17, 20, 26, 32, 34, 40, 41\} \). Hence
we have $W(84,k)$ constructed from four circulants for $k \in \{61,73,75,80,81,82\}$. There are sequences of lengths 11, 10, 11, 10 and weight 39 from [17, Table II.2] and so the construction mentioned at the end of the proof of Theorem 2 gives sequences for the $W(84, 78)$. In addition, [37, p. 345] gives a $W(84, 83)$ and a $W(84, 84)$ constructed using four circulants.

There is an $OD(129, 7, 7, 7)$ and so replacing the variables by the back-circular matrix with first row $\{0 + -\}$, and the circular matrices with first rows $\{- + +\}, \{0 + +\}$ and $\{0 ++\}$ respectively gives a $W(84, 77)$. □

**Lemma 23** There exists a $W(92,k)$ for all $k$ except possibly 71, 72, 75, 77, 79, 83, 85, 87, 89 which are undecided. All are constructed by using four circulant matrices in the Goethals-Seidel array.

**Proof.** We have the result from our Table 8 and Theorem 3 $W(92,k)$ $k \in \{1, \ldots, 70, 72, 74, 76, 78, 80, \ldots, 82, 84, 85, 86, 88, 90, 92\}$. The skew-Hadamard matrix in [37] gives the required $W(92,91)$.

**Lemma 24** There exists a $W(100,k)$ for all $k$ except possibly 71, 73, 77, 81, 91, 93, 95, 97 which are undecided. All are constructed by using four circulant matrices in the Goethals-Seidel array.

**Proof.** The proof is as for Lemma 23 but now the appropriate skew-Hadamard matrix gives a $W(100,99)$.

8 A new construction for weighing matrices

Seberry and W=headers [36, 37] showed that for many orders there are $n+1$ sets of regular matrices $A_1, A_2, \ldots, A_{n+1}$, with entries $1, -1$ and order $n^2$, which pairwise satisfy $A_i A_j^T = \alpha I$, $i \neq j$, a constant, and $\Sigma A_i A_i^T = n^2(n+1)/4$. Using the bipartite graph technique of Rodger, Sarvate and Seberry [33] we can ensure any weighing matrix $W(m,k)$ can be coloured with $k$ colours so that each colour is attached to one and only one non-zero element in each row and column. Thus we have, replacing each element coloured $\pi(i)$ by $A_i \{A_i\}$ and each zero by a zero matrix of order $n^k$.

**Lemma 25** Let $n \equiv \beta(\mod 4)$ be a prime power. Suppose there exists a $W(m, n+1)$. Then there is a $W(m, n^\beta, n^\beta(n+1))$.

**Proof.** Since the $+1$ and the $-1$ cancel each other when the inner product is taken of any two rows or columns of a weighing matrix, the $\alpha I$ contributions from the matrices will also cancel. □

We note that B. Craig [5, 7] has proved related results using isoclinic $k$-sets and systems of distinct representatives.

**Corollary 3** Since a $W(m,4)$ exists for all $m \geq 4$, $m \not= 5, 9$, there exists a $W(9m,36)$ for all $m \geq 4$, $m \not= 5, 9$.

**Corollary 4** Since a $W(2m,5)$ exists for all $m \geq 3$, there exists a $W(18m,45)$ for all $m \geq 3$. 123
Corollary 5. Since a $W(2m, 8)$ exists for all $m \geq 5$, there exists a $W(98m, 392)$ for all $m \geq 5$.

Now corollary 4 gives $W(m, 392)$ for $m = 490, 686$ and $892$. The circulant $W(57, 49)$ can be used to give a $W(114, 96)$ and then taking the Kronecker product with a $W(7, 4)$ we obtain a $W(798, 392)$. Combining these results with those of corollary 2 gives:

Corollary 8. All $W(n, 392)$, $n \equiv 0 \pmod{2}$, $n \geq 880$ exist and they also exist for $n = 490, 686$ and $798$.

Note. We note the circulant $W(57, 49)$ also gives a $W(114, 100)$.

9 New results on the $W(n, 45)$

Since a $W(n, k)$ can only exist for $n$ odd when $k$ is a square we know a $W(n, 45)$ can only exist for even $k$.

Lemma 20. There exists a $W(n, 45)$ for

(i) all $m \equiv 0 \pmod{4}$, $m \geq 48$,

(ii) all $m \equiv 2 \pmod{4}$, $m \geq 48$ and $m = 48, 54$ and $78$.

The unresolved cases are $m \in \{50, 58, 62, 66, 70, 74, 82, 86\}$.

Proof. Ceramiha and Seberry [17, pp346-7] give the existence of $W(n, 45)$ for $n \equiv 0 \pmod{8}$, $n \geq 11$, and $n \in \{52, 60, 68, 76, 84\}$. This combined with the $W(46, 45)$ found by Mathon [31] gives (1) of the enunciation and all $W(2m, 45)$, $m \geq 46$. Corollary 2 gives $W(n, 45)$, $n \in \{54, 90, 126\}$ and the Kronecker product of a $W(6, 5)$ and a $W(13, 9)$ gives a $W(78, 45)$. This completes the proof. □

References


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