Some Orthogonal Designs and complex Hadamard matrices by using two Hadamard matrices

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Abstract

We prove that if there exist Hadamard matrices of order \( h \) and \( n \) divisible by 4 then there exist two disjoint \( W(\frac{1}{2}hn, \frac{1}{2}hn) \), whose sum is a \((1, -1)\) matrix and a complex Hadamard matrix of order \( \frac{1}{2}hn \), furthermore, if there exists an \( OD(m; s_1, s_2, \ldots, s_t) \) for even \( m \) then there exists an \( OD(\frac{1}{2}hm; \frac{1}{2}hns_1, \frac{1}{2}hns_2, \ldots, \frac{1}{2}hns_t) \).

1 Introduction and Basic Definitions

A complex Hadamard matrix (see [4]), say \( C \), of order \( e \) is a matrix with elements \( 1, -1, i, -i \) satisfying \( C^* = ci \), where \( C^* \) is the Hermitian conjugate of \( C \). From [4], any complex Hadamard matrix has order 1 or order divisible by 2. Let \( C = X + iY \), where \( X, Y \) consist of \( 1, -1, 0 \) and \( X \Lambda Y = 0 \) where \( \Lambda \) is the Hadamard product. Clearly, if \( C \) is an complex Hadamard matrix then \( XX^T + YY^T = ci, XY^T = YX^T \).

A weighing matrix [2] of order \( n \) with weight \( k \), denoted by \( W = W(n, k) \), is a \((1, -1, 0)\) matrix satisfying \( WW^T = kf_n \). \( W(n, n) \) is an Hadamard matrix.

Let \( A_j \) be a \((1, -1, 0)\) matrix of order \( m \) and \( A_j A_j^T = s_j I_m \). An orthogonal design \( D = x_1 A_1 + x_2 A_2 + \cdots + x_t A_t \) of order \( m \) and type \((s_1, s_2, \ldots, s_t)\), written \( OD(m; s_1, s_2, \ldots, s_t) \), on the commuting variables \( x_1, x_2, \ldots, x_t \) is a square matrix with entries \( 0, \pm x_1, \pm x_2, \ldots, \pm x_t \), where \( x_i \) or \(-x_i \) occurs \( s_t \) times in each row and column and distinct rows are formally orthogonal. That is

\[
DD^T = \sum_{j=1}^{t} s_j x_j^2 I_m
\]

Let $M$ be a matrix of order $tm$. Then $M$ can be expressed as

$$M = \begin{bmatrix}
M_{11} & M_{12} & \cdots & M_{1t} \\
M_{21} & M_{22} & \cdots & M_{2t} \\
\vdots & & & \vdots \\
M_{t1} & M_{t2} & & M_{tt}
\end{bmatrix}$$

where $M_{ij}$ is of order $m$ ($i, j = 1, 2, \ldots, t$). Analogously with Seberry and Yamada [3], we call this a $t^2$ block $M$-structure when $M$ is an orthogonal matrix.

To emphasize the block structure, we use the notation $M(t)$, where $M(t) = M$ but in the form of $t^2$ blocks, each of which has order $m$.

Let $N$ be a matrix of order $tn$. Then, write

$$N(t) = \begin{bmatrix}
N_{11} & N_{12} & \cdots & N_{1t} \\
N_{21} & N_{22} & \cdots & N_{2t} \\
\vdots & & & \vdots \\
N_{t1} & N_{t2} & & N_{tt}
\end{bmatrix}$$

where $N_{ij}$ is of order $n$ ($i, j = 1, 2, \ldots, t$).

We now define the operation $\circ$ as the following:

$$M(t) \circ N(t) = \begin{bmatrix}
L_{11} & L_{12} & \cdots & L_{1t} \\
L_{21} & L_{22} & \cdots & L_{2t} \\
\vdots & & & \vdots \\
L_{t1} & L_{t2} & & L_{tt}
\end{bmatrix}$$

where $M_{ij}, N_{ij}$ and $L_{ij}$ are of order of $m, n$ and $mn$, respectively and

$$L_{ij} = M_{i1} \times N_{1j} + M_{i2} \times N_{2j} + \cdots + M_{it} \times N_{tj},$$

$i, j = 1, 2, \ldots, t$. We call this the strong Kronecker multiplication of two matrices.

2 Preliminaries

Theorem 1 Let $A$ be an $OD(tm; p_1, \ldots, p_t)$ with entries $z_1, \ldots, z_t$ and $B$ be an $OD(tm; q_1, \ldots, q_t)$ with entries $y_1, \ldots, y_t$, then

$$(A(t) \circ B(t))(A(t) \circ B(t))^T = \left(\sum_{j=1}^{t} p_j x_j^2\right) \left(\sum_{j=1}^{t} q_j y_j^2\right) I_{mn}.$$ 

$(A(t) \circ B(t)$ is not an orthogonal design but an orthogonal matrix.)
Proof.

\[ A(t) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1t} \\ A_{21} & A_{22} & \cdots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nt} \end{bmatrix} \]

and

\[ B(t) = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1t} \\ B_{21} & B_{22} & \cdots & B_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nt} \end{bmatrix} \]

where \( A_{ij} \) and \( B_{ij} \) are of orders \( m \) and \( n \) respectively \((i, j = 1, 2, \cdots, t)\).

Write

\[ C = (A(0) \circ B(0))(A(1) \circ B(1))^T = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1t} \\ C_{21} & C_{22} & \cdots & C_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nt} \end{bmatrix} \]

where \( C_{ij} \) is of order \( mn \).

We first prove \( C_{13} = 0 \). It is easy to calculate \( C_{13} = \)

\[
= \sum_{j=1}^{t} (A_{11} \times B_{1j}) + A_{12} \times B_{2j} + \cdots + A_{1t} \times B_{tj} (A_{31}^T \times B_{1j}^T + A_{32}^T \times B_{2j}^T + \cdots + A_{3t}^T \times B_{tj}^T)
\]

\[
= \sum_{j=1}^{t} ((A_{11} A_{31}^T) \times B_{1j}^T) + (A_{12} A_{32}^T) \times B_{2j}^T + \cdots + (A_{1t} A_{3t}^T) \times B_{tj}^T]
\]

\[
= (A_{11} A_{31}^T + A_{12} A_{32}^T + \cdots + A_{1t} A_{3t}^T) \times \left( \sum_{j=1}^{t} q_j y_j^2 \right) I_n.
\]

But

\[ A_{11} A_{31}^T + A_{12} A_{32}^T + \cdots + A_{1t} A_{3t}^T = 0,
\]

so

\[ C_{13} = 0.
\]

Similarly,

\[ C_{ij} = 0 \; (i \neq j).
\]

We now calculate \( C_{ii} \).

\[
C_{ii} = \sum_{j=1}^{t} (A_{1i} \times B_{1j}) + A_{2i} \times B_{2j} + \cdots + A_{ti} \times B_{tj} (A_{ii}^T \times B_{1j}^T + A_{ii}^T \times B_{2j}^T + \cdots + A_{ii}^T \times B_{tj}^T)
\]

\[
= \sum_{j=1}^{t} ((A_{1i} A_{ii}^T) \times B_{1j}^T) + (A_{2i} A_{ii}^T) \times B_{2j}^T + \cdots + (A_{ti} A_{ii}^T) \times B_{tj}^T]
\]

\[
= (A_{1i} A_{ii}^T + A_{2i} A_{ii}^T + \cdots + A_{ti} A_{ii}^T) \times \left( \sum_{j=1}^{t} q_j y_j^2 \right) I_n.
\]

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\[
= \left( \sum_{j=1}^{l} p_j x_j^2 \right) I_m \times \left( \sum_{j=1}^{l} q_j y_j^2 \right) I_n
\]

\[
= \left( \sum_{j=1}^{l} p_j x_j^2 \right) \left( \sum_{j=1}^{l} q_j y_j^2 \right) I_{mn}.
\]

Thus

\[
(A_t \circ B_t)(A_t \circ B_t)^T = \left( \sum_{j=1}^{l} p_j x_j^2 \right) \left( \sum_{j=1}^{l} q_j y_j^2 \right) I_{mn}.
\]

**Corollary 2** Let A and B be the matrices of orders tm and tn respectively, consist of
1, -1, 0 satisfying AA^T = pI_{mt} and BB^T = qI_{tn}. Then

\[
(A_t \circ B_t)(A_t \circ B_t)^T = pqI_{mn}.
\]

**Proof.** In this case, A = OD(tm; p), B = OD(tn; q) and x_1 = y_1 = 1.

In the remainder of this paper let $H = (H_{ij})$ and $N = (N_{ij})$ of order $h$ and $n$ respectively be 16 block M-structures [3]. So

\[
H = \begin{bmatrix}
H_{11} & H_{12} & H_{13} & H_{14} \\
H_{21} & H_{22} & H_{23} & H_{24} \\
H_{31} & H_{32} & H_{33} & H_{34} \\
H_{41} & H_{42} & H_{43} & H_{44}
\end{bmatrix}
\]

where

\[
\sum_{j=1}^{4} H_{ij}H_{ij}^T = hI_h = \sum_{j=1}^{4} H_{ji}H_{ji}^T,
\]

for $i = 1, 2, 3, 4$ and

\[
\sum_{j=1}^{4} H_{ij}H_{kj} = 0 = \sum_{j=1}^{4} H_{ji}H_{kj},
\]

for $i \neq k, i, k = 1, 2, 3, 4$.

Similarly, let

\[
N = \begin{bmatrix}
N_{11} & N_{12} & N_{13} & N_{14} \\
N_{21} & N_{22} & N_{23} & N_{24} \\
N_{31} & N_{32} & N_{33} & N_{34} \\
N_{41} & N_{42} & N_{43} & N_{44}
\end{bmatrix}
\]

where

\[
\sum_{j=1}^{4} N_{ij}N_{ij}^T = nI_n = \sum_{j=1}^{4} N_{ji}N_{ji}^T,
\]

for $i = 1, 2, 3, 4$ and

\[
\sum_{j=1}^{4} N_{ij}N_{kj} = 0 = \sum_{j=1}^{4} N_{ji}N_{kj},
\]

for $i \neq k, i, k = 1, 2, 3, 4$.
for \( i \neq k, i, k = 1, 2, 3, 4 \).

For ease of writing we define \( X_i = \frac{1}{2}(H_{i1} + H_{i2}), \ Y_i = \frac{1}{2}(H_{i1} - H_{i2}), \ Z_i = \frac{1}{2}(H_{i3} + H_{i4}), \ W_i = \frac{1}{2}(H_{i3} - H_{i4}) \), where \( i = 1, 2, 3, 4 \). Then both \( X_i \pm Y_i \) and \( Z_i \pm W_i \) are \((1, -1)\)-matrices with \( X_i \land Y_i = 0 \) and \( Z_i \land W_i = 0 \), \( \land \) the Hadamard product.

Let
\[
S = \frac{1}{2} \begin{bmatrix} H_{11} + H_{12} & -H_{11} + H_{12} & H_{13} + H_{14} & -H_{13} + H_{14} \\ H_{21} + H_{22} & -H_{21} + H_{22} & H_{23} + H_{24} & -H_{23} + H_{24} \\ H_{31} + H_{32} & -H_{31} + H_{32} & H_{33} + H_{34} & -H_{33} + H_{34} \\ H_{41} + H_{42} & -H_{41} + H_{42} & H_{43} + H_{44} & -H_{43} + H_{44} \end{bmatrix}
\]

Then \( S \) can be rewritten as
\[
S = \frac{1}{2} \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & +1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & +1 \end{bmatrix}
\]
or
\[
S = \begin{bmatrix} X_1 & -Y_1 & Z_1 & -W_1 \\ X_2 & -Y_2 & Z_2 & -W_2 \\ X_3 & -Y_3 & Z_3 & -W_3 \\ X_4 & -Y_4 & Z_4 & -W_4 \end{bmatrix}
\]

Obviously, \( S \) is a \((0, 1, -1)\) matrix.

Write
\[
R = \begin{bmatrix} Y_1 & X_1 & W_1 & Z_1 \\ Y_2 & X_2 & W_2 & Z_2 \\ Y_3 & X_3 & W_3 & Z_3 \\ Y_4 & X_4 & W_4 & Z_4 \end{bmatrix}
\]
also a \((0, 1, -1)\) matrix.

We note \( S \pm R \) is a \((1, -1)\) matrix, \( R \land S = 0 \) and by Corollary 1
\[
SS^T = RR^T = \frac{1}{2} h I_h,
\]

**Lemma 3** If there exists an Hadamard matrix of order \( h \) divisible by 4, there exists an \( OD(h; \frac{1}{2} h, \frac{1}{2} h) \).

**Proof.** From \( S \) and \( R \) as above, \( H = S + R \). Note \( HH^T = SS^T + RR^T + SR^T + RS^T = h I_h \) and \( SS^T = RR^T = \frac{1}{2} h I_h \), hence \( SR^T + RS^T = 0 \). Let \( x \) and \( y \) be commuting variables then \( E = xS + yR \) is the required orthogonal design.
3 Weighing Matrices

Lemma 4 If there exist Hadamard matrices of order $h$ and $n$ divisible by 4, there exists a $W(\frac{1}{4}hn, \frac{1}{4}hn)$.

Proof. Let $H$ and $N$ as above be the Hadamard matrices of order $h$ and $n$ respectively. Let

\[
P = \frac{1}{2} \begin{bmatrix} X_1 & Y_1 & Z_1 & W_1 & \vdots & \vdots \\
X_2 & Y_2 & Z_2 & W_2 \\
X_3 & Y_3 & Z_3 & W_3 \\
X_4 & Y_4 & Z_4 & W_4 \\
\end{bmatrix} \odot \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\
N_{21} & N_{22} & N_{23} & N_{24} \\
N_{31} & N_{32} & N_{33} & N_{34} \\
N_{41} & N_{42} & N_{43} & N_{44} \\
\end{bmatrix}
\]

Rewrite

\[
P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\
P_{21} & P_{22} & P_{23} & P_{24} \\
P_{31} & P_{32} & P_{33} & P_{34} \\
P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}
\]

Consider

\[
P_{11} = \frac{1}{2}(X_1 \times N_{11} + Y_1 \times N_{21} + Z_1 \times N_{31} + W_1 \times N_{41}),
\]

where both $X_1 \times N_{11} + Y_1 \times N_{21}$ and $Z_1 \times N_{31} + W_1 \times N_{41}$ are $(1,-1)$ matrices. So $P_{11}$ has entries 1, -1, 0 and similarly for other $P_{ij}$. By Lemma 1,

\[
P P^T = \frac{1}{8}hnI_{\frac{1}{4}hn}.
\]

Then $P$ is a $W(\frac{1}{4}hn, \frac{1}{4}hn)$.

Corollary 5 There exists a $W(h, \frac{1}{2}h)$ ($h > 1$) if there exists an Hadamard matrix of order $h$.

Proof. If $h > 2$ let $n = 4$ in Theorem 1. For the case $h = 2$, note $W(2,1)$ is the identity matrix.

We also note that if

\[
Q = \frac{1}{2} \begin{bmatrix} X_1 & Y_1 & Z_1 & W_1 \\
X_2 & Y_2 & Z_2 & W_2 \\
X_3 & Y_3 & Z_3 & W_3 \\
X_4 & Y_4 & Z_4 & W_4 \\
\end{bmatrix} \odot \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\
N_{21} & N_{22} & N_{23} & N_{24} \\
N_{31} & N_{32} & N_{33} & N_{34} \\
N_{41} & N_{42} & N_{43} & N_{44} \\
\end{bmatrix}
\]

Then $Q$ is also a $W(\frac{1}{4}hn, \frac{1}{4}hn)$.

Theorem 6 Suppose $h$ and $n$ divisible by 4, the orders of Hadamard matrices then there exist two disjoint $W(\frac{1}{4}hn, \frac{1}{4}hn)$, whose sum and difference are $(1,-1)$ matrices.
Rewrite

\[ Q = \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & Q_{14} \\
Q_{21} & Q_{22} & Q_{23} & Q_{24} \\
Q_{31} & Q_{32} & Q_{33} & Q_{34} \\
Q_{41} & Q_{42} & Q_{43} & Q_{44}
\end{bmatrix} . \]

We note

\[ P_{ij} = \frac{1}{2}(X_i \times N_{ij} + Y_i \times N_{2j} + Z_i \times N_{3j} + W_i \times N_{4j}) , \]

and

\[ Q_{ij} = \frac{1}{2}(X_i \times N_{ij} + Y_i \times N_{2j} - Z_i \times N_{3j} - W_i \times N_{4j}) . \]

Since \( P_{ij} + Q_{ij} = X_i \times N_{ij} + Y_i \times N_{2j} \) and \( P_{ij} - Q_{ij} = Z_i \times N_{3j} + W_i \times N_{4j} \) we conclude that \( P_{ij} \pm Q_{ij} \) are \((1,-1)\) matrices and \( P_{ij} \wedge Q_{ij} = 0 \). Thus \( P \pm Q \) is a \((1,-1)\) matrix and \( P \wedge Q = 0 \). \( P \) and \( Q \) are both \((\frac{1}{2}hn, \frac{1}{2}hn)\) by Corollary 1.

4 Complex Hadamard Matrices

Lemma 7 \( PQ^T = QP^T \).

Proof. Write

\[ PQ^T = \begin{bmatrix}
E_{11} & E_{12} & E_{13} & E_{14} \\
E_{21} & E_{22} & E_{23} & E_{24} \\
E_{31} & E_{32} & E_{33} & E_{34} \\
E_{41} & E_{42} & E_{43} & E_{44}
\end{bmatrix} . \]

and

\[ QP^T = \begin{bmatrix}
F_{11} & F_{12} & F_{13} & F_{14} \\
F_{21} & F_{22} & F_{23} & F_{24} \\
F_{31} & F_{32} & F_{33} & F_{34} \\
F_{41} & F_{42} & F_{43} & F_{44}
\end{bmatrix} . \]

We first prove \( E_{13} = F_{13} \).

We note

\[ E_{13} = \]

\[ = \frac{1}{4} \sum_{j=1}^{4} (X_i \times N_{1j} + Y_i \times N_{2j} + Z_i \times N_{3j} + W_i \times N_{4j})(X_j^T \times N_{1j}^T + Y_j^T \times N_{2j}^T - Z_j^T \times N_{3j}^T - W_j^T \times N_{4j}^T) \]

and

\[ F_{13} = \]

\[ = \frac{1}{4} \sum_{j=1}^{4} (X_i \times N_{1j} + Y_i \times N_{2j} - Z_i \times N_{3j} - W_i \times N_{4j})(X_j^T \times N_{1j}^T + Y_j^T \times N_{2j}^T + Z_j^T \times N_{3j}^T + W_j^T \times N_{4j}^T) . \]
Obviously, \( E_{13} = F_{13} \) if and only if
\[
\sum_{j=1}^{4} (X_j \times N_{1j} + Y_j \times N_{2j})(Z_j^T \times N_{3j}^T + W_j^T \times N_{4j}^T) = 0, \tag{1}
\]

or
\[
\sum_{j=1}^{4} (Z_j \times N_{3j} + W_j \times N_{4j})(X_j^T \times N_{1j}^T + Y_j^T \times N_{2j}^T) = 0. \tag{2}
\]

To show this, note
\[
\sum_{j=1}^{4} (X_j \times N_{1j})(Z_j^T \times N_{3j}^T) = \sum_{j=1}^{4} (X_j Z_j^T) \times (N_{1j} N_{3j}^T) = X_1 Z_1^T \times \sum_{j=1}^{4} N_{1j} N_{3j}^T = 0,
\]

and similarly for other parts in (1) and (2). Thus \( E_{13} = F_{13} \). Similarly, \( E_{ij} = F_{ij} \), for other \( i \neq j \).

We now prove \( E_{ui} = F_{ui} \). We see
\[
E_{ui} = \frac{1}{4} \sum_{j=1}^{4} (X_j \times N_{1j} + Y_j \times N_{2j} + Z_j \times N_{3j} + W_j \times N_{4j})(X_j^T \times N_{1j}^T + Y_j^T \times N_{2j}^T + Z_j^T \times N_{3j}^T + W_j^T \times N_{4j}^T)
\]

and
\[
F_{ui} = \frac{1}{4} \sum_{j=1}^{4} (X_j \times N_{1j} + Y_j \times N_{2j} + Z_j \times N_{3j} + W_j \times N_{4j})(X_j^T \times N_{1j}^T + Y_j^T \times N_{2j}^T + Z_j^T \times N_{3j}^T + W_j^T \times N_{4j}^T).
\]

Obviously, \( E_{ui} = F_{ui} \) if and only if
\[
\sum_{j=1}^{4} (X_j \times N_{1j} + Y_j \times N_{2j})(Z_j^T \times N_{3j}^T + W_j^T \times N_{4j}^T) = 0, \tag{3}
\]

or
\[
\sum_{j=1}^{4} (Z_j \times N_{3j} + W_j \times N_{4j})(X_j^T \times N_{1j}^T + Y_j^T \times N_{2j}^T) = 0. \tag{4}
\]

The proof is the same as in (1) and (2). Hence \( E_{ui} = F_{ui} \). Finally, we conclude \( PQ^T = QP^T \).

**Theorem 8** If there exist Hadamard matrices of order \( h \) and \( n \) divisible by 4 then there exists a complex Hadamard matrix of order \( \frac{1}{2}hn \).

**Proof.** By the proof of Theorem 2, \( P \) and \( Q \) are the two disjoint \( W(\frac{1}{2}hn, \frac{1}{2}hn) \) i.e. \( P \land Q = 0 \)

and \( P \pm Q \) is a \((1, -1)\) matrix. Furthermore by Lemma 3, \( PQ^T = QP^T \). Thus \( P + iQ \) is a complex Hadamard matrix of order \( \frac{1}{2}hn \).

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5 Orthogonal Designs

Theorem 9 If there exist Hadamard matrices of order $h$, $n$ divisible by 4 and an $OD(m; s_1, s_2, \ldots, s_l)$, where $m$ is even, then there exists an

$$OD(\frac{1}{4}hnm; \frac{1}{4}hns_1, \frac{1}{4}hns_2, \ldots, \frac{1}{4}hns_l).$$

Proof. Let

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix},$$

be the $OD(m; s_1, s_2, \ldots, s_l)$ on the commuting variables $x_1, \ldots, x_l$, where $D_j$ is of order $\frac{1}{2}m$. Let

$$D' = \begin{bmatrix} P & Q \\ -Q & P \end{bmatrix} \circ \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$$

where $P$ and $Q$, constructed above, are from the Hadamard matrices of order $h$ and $n$.

Then by Theorem 3 and Corollary 1,

$$D'D'^T = \frac{1}{4}hn(\sum_{j} s_j x_j^2) H_{\frac{1}{4}hn}.$$ 

Since $P \land Q = 0$, if $D$ consists of $0, \pm x_1, \ldots, \pm x_l$ then $D'$ also consists of $0, \pm x_1, \ldots, \pm x_l$ so $D'$ is an $OD(\frac{1}{4}hnm; \frac{1}{4}hns_1, \frac{1}{4}hns_2, \ldots, \frac{1}{4}hns_l)$.

Corollary 10 If there exist Hadamard matrices of order $h$ and $n$ divisible by 4 then there exists an $OD(\frac{1}{2}hn; \frac{1}{4}hn, \frac{1}{4}hn)$.

Proof. Let

$$D = \begin{bmatrix} z & y \\ -y & z \end{bmatrix}$$

in the proof of Theorem 4, where $x$ and $y$ are commuting variables, put $m = l = 2$ and $s_1 = s_2 = 1$.

6 Remark

Theorem 1 cannot be replaced by Corollary 1 because the existence of Hadamard matrices of order $h$ and $n$ does not imply the existence of an Hadamard matrix of order $\frac{1}{4}hn$. For example, there exist Hadamard matrices of order 4·3 and 4·71 but no Hadamard matrix of order 4·213 has been found [1], however, by Theorem 1, we have a $W(4·213, 2·213)$.
By the same result, there exists a $W(4k, 2k)$ and a complex Hadamard matrix of order $4k$, where $k$ is

$$
\begin{array}{cccccccccccc}
781 & 789 & 917 & 1315 & 1349 & 1441 & 1633 & 1703 & 2059 & 2227 & 2489 & 2515 \\
2627 & 2733 & 3013 & 3273 & 3453 & 3479 & 3715 & 4061 & 4331 & 4435 & 4757 & 4781 \\
4899 & 4979 & 4997 & 5001 & 5109 & 5371 & 5433 & 5467 & 5515 & 5533 & 5609 & 5755 \\
5767 & 5793 & 5893 & 6009 & 6059 & 6177 & 6209 & 6233 & 6333 & 6377 & 6497 & 6539 \\
6801 & 6881 & 6887 & 6943 & 7233 & 7277 & 7387 & 7513 & 7555 & 7663 & 7739 & 7811 \\
7999 & 8023 & 8057 & 8189 & 8549 & 8591 & 8611 & 8633 & 8809 & 8879 & 8927 & 9055 \\
9097 & 9167 & 9557 & 9563 & 9573 & 9659 & 9727 & 9753 & 9757 & 9869 & 9913 & 9991 \\
\end{array}
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References


