Maximal q-ary Codes and Plotkin’s Bound

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Abstract

The analogue of Plotkin’s bound for q-ary codes with high distance relative to length was given by Blake and Mullin as

\[ A(n,d) \leq \frac{qd}{qd - n(q-1)}, \quad qd > (q - 1)n. \]

Further we show

\[ A(n,(q - 1)n/q) = qd, qd = (q - 1)n. \]

Generalized Hadamard matrices are used to obtain q-ary codes which meet these bounds. The q-ary analogue of Levenshtein’s construction is discussed and maximal codes constructed.

The codes given are often maximum distance separable, and constructions are given which include the Reed-Solomon codes, but exist for cases when the Reed-Solomon codes cannot exist.

We also study block codes over non-binary alphabets which may prove fruitful for multiple channel encoding.

1 The Plotkin Bound

By counting the sum

\[ \sum_{u \in G} \sum_{v \in G} \text{dist}(u,v) \]

in two ways Blake and Mullin (p.86) show that

**THEOREM 1** For an alphabet of q symbols, the maximum number of codewords of length n and distance d, A(n,d), is

\[ A(n,d) \leq \frac{qd}{qd - (q - 1)n} \quad \text{for } qd > (q - 1)n \geq (q - 1)d. \]

The maximum occurs when each symbol occurs A(n,d/q) times in each column which requires q\|A(n,d).

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**Lemma 2** (i) $A(n, d) \leq qA(n-1, d)$; (ii) $A(qn, (q-1)n) \leq q^n n$.

**Proof.** Given a q-ary code, the codewords fall into q classes, those beginning with 0, 1, ..., $(q-1)$. One class must contain at least $1/q$ of the codewords, and so

$$A(n-1, d) \geq A(n, d)/q.$$ 

Thus, using theorem 1 we have

$$A(qn, (q-1)n) \leq qA(qn-1, (q-1)n)$$

$$\leq q \left( \frac{q(q-1)n}{q(q-1)n - (q-1)(qn-1)} \right)$$

$$= q^n n.$$

We now use generalized Hadamard matrices over groups of size $q, GH(n, G)$, to obtain codes which meet these bounds. Although we have discussed the case $q = 3$ elsewhere Table 1 revises the knowledge there. This paper considers $q \neq 2, 3$.

Remark. We should like to acknowledge that Christiane Engelmann and Michael Kammengarn of West Germany have pointed out that our use in a previous paper of

$$A(n, d) \leq 3 \left[ \frac{d}{3d - 2n} \right] \text{ for } 3d > 2n \geq 2d$$

is incorrect and the correct result is

$$A(n, d) \leq \frac{3d}{3d - 2n} \text{ for } 3d > 2n \geq 2d.$$

A square matrix of size $n$ with entries from a group $G$ is called a **generalized Hadamard matrix**, $GH(n, G)$, if the inner product of any two distinct rows, $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$, $a_i, b_i \in G$, defined by $a \cdot b = \sum_{i=1}^{n} a_i b_i^{-1}$ is $n/|G|$ copies of $G$. For example, we have

$$GH(5, Z_5) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & a & a^2 & a^3 & a^4 \\
a^4 & a^3 & a & a^2 \\
a^2 & a & a^4 & a^3 \\
a & a^2 & a^3 & a^4 \\
\end{bmatrix}$$

$$GH(6, Z_5) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & w & w^2 & w^3 & w^4 \\
w & 1 & w & w^2 & w^3 \\
w^2 & w & 1 & w & w^3 \\
w^3 & w^2 & w & 1 & w \\
w^4 & w^3 & w^2 & w & 1 \\
\end{bmatrix}$$

38
Our constructions will extend the first of these into a code equivalent to a Reed-Solomon code. However our constructions applied to the second example give a maximal code, similar to those of Reed-Solomon, but where Reed-Solomon codes cannot exist because the finite field does not exist, for example for \( q = 6 \).

The results of de Launey, Drake, Jungnickel, Rajkundlia, Seberry, Seiden, Street and Dawson allow us to say:

**Lemma 3** Let \( EA(p^i) \) be the elementary abelian group of order \( p^i \), where \( p \) is a prime. Then the following generalized Hadamard matrices exist:

(i) \( GH(p^{i+j}, EA(p^i)) \) for all \( i \geq 1, j \geq 0 \);

(ii) \( GH(2^{i+j}p^{m}, EA(p^i)) \) for all \( 0 \leq k + j \leq m, 0 \leq k, j \leq m, m \geq 1, i \geq 1 \);

(iii) \( GH(2p^i, EA(p^j)), GH(4p^i, EA(p^j)) \) for all \( 1 \geq 1 \);

If \( p^i - 1 = r^s \) for some prime \( r \), then there exists:

(iv) \( GH(p^{i+h}, EA(p^i)) \) for all \( 1 \leq i \leq t, 1 \leq j \leq k, \ell \geq i \) or \( \ell = 0 \);

(v) \( GH(2^{i+j}p^{m-r+n}, EA(p^i)) \) for all \( 0 \leq i + j + k \leq m, 0 \leq i, j, k \leq m, m, r, t \geq 1 \);

If \( q \) is a prime power and there exists a \( GH(q+1; G) \) for some group \( G \), then there exists:

(vi) \( GH(q^i(q + 1); G) \) for all \( i \geq 1 \).

For example there exists a \( GH(0; Z_3) \) so there exists \( GH(3^i, 0; Z_3) \).

As well de Launey [1984] surveys the current knowledge on non existence of generalized Hadamard matrices.

Any \( GH(n, G) \) is equivalent to a \( GH(n, G) \) with its first row and column consisting entirely of the unit element of the group.

**Lemma 4** A \( GH(n, G), |G| = q \), gives block codes over a \( q \)-symbol alphabet with parameters, \( (n, M, d) \):

(i) \( (n, qn, (q - 1)n/q) \),

(ii) \( (n - 1, qn, (q - 1)n/q - 1) \),

(iii) \( (n - 1, n, (q - 1)n/q) \),

(iv) \( (n - 2, n, (q - 1)n/q - 1) \),

(v) \( (q + 1, q^2, q) \).
(i), (iii), (v) are maximal.

**Proof.** Write $A$ for the normalized $GH(n, G)$ and $G = \{ e = a_1, \ldots, a_q \}$.
Then the required codes are:

(i) 

$$B = \begin{pmatrix} A \\ a_2A \\ \vdots \\ a_qA \end{pmatrix}$$

where $a_iA$ has the usual meaning of multiplying every element of $A$ by $a_i$;

(ii) $B$ with any column removed;

(iii) $A$ with the first column removed;

(iv) $A$ with the first and any other column removed;

(v) where $n = q$, let $c$ be any column of $A$ except the first, then $C$ is the code

$$C = \begin{pmatrix} Ac \\ a_2Ac \\ \vdots \\ a_qAc \end{pmatrix}$$

The result follows as the distance of any two rows of $A$ is $(q - 1)$.

Remark. An interesting paper of Zlotnik deals similarly with extended Reed-Solomon codes but his/her results are for different alphabets.

A number of authors, including de Launey, Lam, Seberry, and Street and Rodger, have studied an extension of generalized Hadamard matrices in which the elements are over a group ring, called Bhaskar Rao designs (BRD). We consider the group ring $\{0\} + G$. A *generalized Bhaskar Rao design* (GBRD) $W = g_1A_1 + g_2A_2 + \ldots + g_rA_r$, $g_1G$, with parameters $v, b, r, k, \lambda$ satisfies

$$WW^T = rI_v$$

$$\left( \sum_{i=1}^r A_i \right) \left( \sum_{i=1}^r A_i^T \right) = (r - \lambda)I + \lambda J$$

$$J\sum_{i=1}^r A_i = kJ$$
\[
\left( \sum_{i=1}^{r} A_i \right) J = rJ,
\]

where \( A_i \) are \((0,1)-matrices\), \( \sum_{i=1}^{r} A_i \) is a BIBD \((v, b, r, k, \lambda)\). The GBRD is written GBRD\((v, b, r, k, \lambda; G)\) or GVRD\((v, k, \lambda)\) for brevity. Such designs can be extended to partially balanced and pairwise balanced designs.

In the remainder of this section we use

\[
r = \lambda(v - 1)/(k - 1), \quad b = vr/k.
\]

**Lemma 5** If there exists a GBRD\((v, k, \lambda; G)\), with \(|G| = q - 1\) then writing \( t = 2(r - \lambda) + (q - 2)\lambda/(q - 1) \) there exists \(q\)-ary codes with parameters

(i) \((b, v, t)\),

(ii) \((b, v + q, \min(t, r, b - r))\),

(iii) \((b, qv, \min(r, t))\).

**Proof.** Let \(M\) be the BRD and \(g_1, g_2, \ldots, g_{q-1}\) the group elements. The result follows by considering the rows of

\[
M = \begin{bmatrix}
0 & \cdots & 0 \\
g_1 & \cdots & g_1 \\
\vdots & \ddots & \vdots \\
g_{q-1} & \cdots & g_{q-1} \\
\vdots & \cdots & M \\
M & \cdots & M
\end{bmatrix}, \quad \begin{bmatrix}
M \\
g_1M \\
\vdots \\
g_{q-1}M
\end{bmatrix},
\]

respectively as codewords.

**Remark.** Codes from BRD over alphabets other than binary should be explored as the zero-nonzero coordinates provide a code in themselves for error correction at one rate while the nonzero coordinates provide a non-binary code with maximum distance separable or near maximum distance separable codewords which could be exploited using phase or frequency variations.

### 2 A property of \(q\)-ary codes used to give more codewords

**Lemma 6** Let \(a\) and \(b\) be two \(q\)-ary vectors. Then with \(p = q - 1\)
\[
\sum_{i=0}^{p} d(a, b + 1) = 2n
\]
\[
d(a + i, b + j) = d(a + i + k, b + j + k), \quad i, j, k \in \{0, 1, \ldots, p\}.
\]
where \(d\) is the Hamming distance.

**Proof.** The second part of the lemma is obviously true for linear codes but we show it is also true for block codes. We write the two codewords as

\[
a = 0 \quad \ldots \quad 0 \quad \ldots \quad p \quad \ldots \quad 0
\]
\[
b = 0 \quad \ldots \quad 0 \quad \ldots \quad p \quad \ldots \quad 0 \quad \ldots \quad 0 \quad \ldots \quad p
\]
\[
\begin{array}{c}
x_{i0} \quad x_{ip} \\
x_{p0} \\
x_{pp}
\end{array}
\]

\(x_{ij}\) is the number of coordinates which are \(i\) in \(a\) and \(j\) in \(b\).

Now

\[
d(a, b) = d(a + i, b + i) = n - \sum_{j=0}^{p} x_{ij}
\]
\[
d(a + i, b) = d(a + i, b + i - i) = n - \sum_{j=0}^{p} x_{ij+j}
\]

\vdots

\[
d(a + k, b) = d(a + i, b + i - k) = n - \sum_{j=0}^{p} x_{ij+j+k}
\]

Further,

\[
\sum_{k=0}^{p} d(a, b + k) = q \sum_{i=0}^{p} x_{ij} = (q - 1)n.
\]

This allows us to readily test the distance of a constructed code, as in the following:

**Lemma 7** Suppose \(A\) is a \(q\)-ary \((p = q - 1)\) \((n, M, d)\) code. Then, writing \(A + i\),
to denote adding \(i\) to each element of \(A\) (assumed written on an additively defined alphabet).

\[
\begin{bmatrix}
A \\
A + 1 \\
\vdots \\
A + p
\end{bmatrix}
\]

42
is a q-ary \( (n, q, M, d') \)-code where
\[
d' = \min \left[ d(a, b), d(a, b + 1), \ldots, d(a, b + p - 1), (q - 1)n - \sum_{j=0}^{p-1} d(a, b + j) \right]
\]
a, b codewords of A.

EXAMPLE 8 Suppose
\[
A = \begin{pmatrix}
4 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 1 & 3 \\
0 & 2 & 4 & 0 & 1 \\
0 & 3 & 2 & 4 & 0 \\
0 & 0 & 1 & 3 & 2 \\
\end{pmatrix}, \quad \text{a 5-ary} \ (6, 6, 5) \ - \text{code},
\]
then to find \( d' \) we merely need to test \( a = (4, 0, 0, 0, 0, 0) \) with \( b = (0, 4, 0, 1, 3, 2) + i \) which gives \( d = \min(4, 5) = 4 \) and \( a = (0, 4, 0, 1, 3, 2) \) with \( b = (0, 2, 4, 0, 1, 3) + i \) which gives \( d = \min(4, 5) = 4 \). This gives \( d' = \min[4, 5, 4, 5] = 4 \) giving a \( (6, 30, 4) \) 5-ary code.

THEOREM 9 Let \( A \) be a q-ary \( (n - 1, M, d) \)-code. Further define
\[
d_i = \min_{a, b \in A} d(a, a + i), \quad i = 0, 1, \ldots, q - 1.
\]
Then there exists q-ary \( (n, qM, \min(d, d_i + 1)) \) and \( (n - 1, qM, \min(d_i)) \) codes.

Proof. We consider, using the notation of Lemma 7,
\[
\left( \begin{array}{c}
0 \\
\vdots \\
0 \\
1 \\
\vdots \\
A + 1 \\
1 \\
\vdots \\
q - 1 \\
\vdots \\
A + q - 1 \\
q - 1 \\
\end{array} \right) = \left( \begin{array}{c}
N_0 \\
N_1 \\
\vdots \\
N_{q-1} \\
\end{array} \right)
\]
This gives the first code. The second follows from Lemma 7.
3 Levenshtein’s method

Let us suppose that an arbitrary $GF(m, C)$, $M = k|C| = kq$ exists, written on
the additive group, whose first column is composed entirely of zero’s; denote this
matrix $M_0$, and the matrix, when formed by stripping the column of zero’s, by
$M^*_n$.

The theory giving the construction of maximal codes requires matrices of par-
ticular orders and distances.

**Lemma 10.** If there exists an $M_{kq}$ (respectively $M_{k(q+1)}$) then the rows of $M'_{kq}$
(respectively $M'_{k(q+1)}$) form a code with parameters $n = kq - 1, M = kq, d + k(q - 1)$
(respectively $n = kq + q - 1, M = q(k + 1), d = (k + 1)(q - 1)$).

Write

$$i = \left\lfloor \frac{d}{qd - n(q - 1)} \right\rfloor.$$

**Lemma 11.** If $qd > (q - 1)n \geq (q - 1)d$, then there exist integers $a$ and $b$ such that

$$\begin{cases}
  n = a(qi - 1) + b(qi + q - 1) \\
  d = (q - 1)ai + (q - 1)b(i + 1).
\end{cases}$$

(1)

**Proof.** We can define $i$ in terms of the following inequalities:

$$\left(\frac{d}{qd - n(q - 1)}\right) - 1 < i \leq \left(\frac{d}{qd - n(q - 1)}\right),$$

that is,

$$\frac{(q - 1)(n - d)}{qd - n(q - 1)} < i \leq \frac{d}{qd - n(q - 1)}.$$

Considering the left inequality gives

$$\frac{(q - 1)(i + 1)}{qi + (q - 1)} < \frac{d}{n},$$

(2)

and the right inequality gives

$$\frac{d}{n} \leq \frac{(q - 1)i}{qi - 1}.$$  

(3)

Combining (2) and (3) we obtain

$$\frac{(q - 1)(i + 1)}{qi + (q - 1)} < \frac{d}{n} \leq \frac{(q - 1)i}{qi - 1}.$$  

(4)
The two inequalities of (4) may be written in determinant form:

\[
\begin{vmatrix}
 d & (q-1)(i+1) \\
 n & qi + (q-1)
\end{vmatrix} > 0 = A, \quad \text{say}
\]

\[
\begin{vmatrix}
 d & (qi - 1) \\
 n & (q - 1)i
\end{vmatrix} \geq 0 = B, \quad \text{say.}
\]

Now suppose that both \(A\) and \(B\) are both divisible by \(q - 1\). Then let

\[
A = (q-1)a
\]

\[
B = (q-1)b
\]

so (5) and (6) become

\[
A = (q-1)a = d(qi + q - 1) - n(q - 1)(i + 1)
\]

\[
B = (q-1)b = (q-1)ni - d(qi - 1),
\]

and solving (7) and (8) for \(n\) and \(d\) yields the required results (1).

Note: Requiring that \(A\) and \(B\) are divisible by \(q - 1\) imposes only one condition, namely that \(d\) is also divisible by \(q - 1\), but in the case of \(q\)-ary codes the distance is, in fact, divisible by \((q - 1)\) for maximal codes as then \(qd = (q - 1)n\).

Following Levenshtein, we define the operation of adjunction of the matrices

\[
X = (X_{ij}), \quad i = 1, 2, \cdots, L_1, \quad j = 1, 2, \cdots, n_1
\]

\[
Y = (Y_{ij}), \quad i = 1, 2, \cdots, L_2, \quad j = 1, 2, \cdots, n_2
\]

as follows:

\[
X + Y = \begin{pmatrix}
X_{11} & X_{12} & \cdots & X_{1n_2} & Y_{11} & Y_{12} & \cdots & Y_{1n_2} \\
X_{21} & X_{22} & \cdots & X_{2n_2} & Y_{21} & Y_{22} & \cdots & Y_{2n_2} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \cdots & \vdots \\
X_{L_1} & X_{L_2} & \cdots & X_{L_2n_2} & Y_{L_1} & Y_{L_2} & \cdots & Y_{L_2n_2}
\end{pmatrix}
\]

where \(L = \min(L_1, L_2)\).

The operation of extension of a matrix \(X\), \(r\) times, is defined as the result of the consecutive adjunction of \(r\) matrices \(X\).

Note: If the rows of the matrix \(X\) form a code with the parameters \(n_1, d_1\) and \(M_1\), and the rows of a matrix \(Y\) form a code with the parameters \(n_2, d_2\) and \(M_2\), then
the rows of the matrix \( aX + bY \), where \( a \) and \( b \) are integer non-negative numbers, form a code with the parameters

\[
\begin{align*}
  n &= an_1 + bn_2 \\
  d &= ad_1 + bd_2
\end{align*}
\]

and

\[
M = \min(M_1, M_2).
\]

**Theorem 12** If \( d \) is divisible by \((q - 1)\) and \( qd > (q - 1)n \geq (q - 1)d \), then the following matrix \( M \) is maximal in that it meets the bound

\[
A(n, d) = q \left\lfloor \frac{d}{qd - (q - 1)n} \right\rfloor = qi
\]

\[
M = aM'_i + bM'_{i+1}
\]

\[
a = \frac{d(qi + q - 1)/(q - 1) - n(i + 1)}{i - d(qi - i)/(q - 1)}
\]

\[
b = \frac{m - d(qi - i)/(q - 1)}{i - d(qi - i)/(q - 1)}.
\]

**Proof.** For the proof of the theorem, it is sufficient to see, using Lemmas 10 and 11, that the above construction does indeed generate a maximal code.

**Example 13** Let

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & i & i^3 \\
1 & i^2 & 0 & 1 & i \\
1 & i & i^3 & 0 & i \\
1 & i & i^3 & i^2 & 0
\end{bmatrix}
\]

be the \( \text{GBRD}(6,5,4;Z_4) \). Then \( A \) is a \((6,6,5)\)-code over a 5-ary alphabet and

\[
B = \begin{bmatrix}
A \\
iA \\
i^2A \\
i^3A
\end{bmatrix}
\]

is a \((6,24,5)\)-code over a 5-ary alphabet and

\[
\begin{bmatrix}
0 & \cdots & 0 \\
B
\end{bmatrix}
\]

is a \((6,25,5)\) maximal 5-ary code.
EXAMPLE 14 From Seberry (1980) we have that a
\[ A = \text{GBRD}(p^t + 1, p^t, p^t - 1; Z_t) \text{ exists whenever } p^t \text{ is a prime power, } t \text{ divides } p^t - 1 \text{ and } Z_t \text{ is a cyclic group. If the elements of the group are } 1, g_1, \ldots, g_{t-1}, \text{ then} \]
\[
\begin{bmatrix}
0 & \cdots & 0 \\
A & g_1A \\
& \cdot \\
g_{t-1}A
\end{bmatrix}
\]
is a \((p^t + 1, t(p^t + 1) + 1, p^t)\)-code over a \((t+1)\)-ary alphabet. When \(t = p^t - 1\) this gives a maximal code.

COROLLARY 15 Let \(p^t\) be a prime power. Then there exists a \((p^t + 1, p^t + 1, p^t)\)
maximal code over a \(p^t\)-ary alphabet.

EXAMPLE 16 From de Launey (1987) a \(W = \text{GBRD}\left(\sum_{j=1}^{t-1} p^t, p^t, p^{t-2}(p-1); Z_{t-1}\right)\)
t \(\geq 3\), exists whenever \(p^t\) is a prime power and \(Z_{t-1}\) is a cyclic group. Then proceeding as in Example 14 we have a \(((p^t - 1)/(p - 1), p^{t-2}(p - 1) + 1, p^{t-2}(p - 1))\)
-code over a \(p\)-ary alphabet.

EXAMPLE 17 The following are \(4\)-ary codes.

A structured \((7,8,6)\)-code (maximal)

- aaaaaa, baceebc
- cebcbac, cbaaceeb
- ceeebcsa, bcbacec
- aceebcb, ebcbace

A structured \((5,10,4)\)-code (maximal)

A \((6,8,5)\)-code

- aaaaa, ceecc, aabdc, abedo
- aceec, aaaaa, daabc, dabac
- cacec, ecabdc, cdaab, cdbbc
- eceac, bceaebc, bcdaa, acdab
- eceed, cbceebc, abcde, bcdea
- cecce, bcebeab, abcde
- abceeb, abceeb
- babcc, eabce
- cbabc, ccbab
- bceba, aebbe
- eceeb, bceeb
- beaeb, bheae
- ebbea

A \((10,5,9)\)-code

- aaaaa, ceecc, aabdc, abedo
- aceec, aaaaa, daabc, dabac
- cacec, ecabdc, cdaab, cdbbc
- eceac, bceaebc, bcdaa, acdab
- eceed, cbceebc, abcde, bcdea
- cecce, bcebeab, abcde
- abceeb, abceeb
- babcc, eabce
- cbabc, ccbab
- bceba, aebbe
- eceeb, bceeb
- beaeb, bheae
- ebbea

47
EXAMPLE 18 The following are 5-ary codes.

5-ary (7,15,8)-code (maximal) 5-ary (8,10,7)-code

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Table 1

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48
### Table 2
Codes with $45 \geq 3n \geq 36$, $q = 4$

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### Table 3
Codes with $51 \geq 4n \geq 48$, $q = 5$

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