VANSTONE'S CONSTRUCTION APPLIED
TO BHASKAR RAO DESIGNS

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ABSTRACT

We show how Vanstone's construction, given in his paper "A note on a construction for BIBD's", Utilitas Mathematica, 7(1975), 321-322, can be applied to symmetric GBRD(v, k, \lambda; | G |), | G | odd, can be used to obtain GBRD(v, \binom{v}{2}, \binom{k}{2}, \lambda, \binom{k}{2}; G) and hence many families of BIBD.

1. INTRODUCTION

Definitions of SBIBD and BIBD are standard.

Let A = \{a_{ij}\} be a matrix of order with a_{ij} \in \{0, 1, -1\}. A is called a weighing matrix of weight p and order n, if AA^T = A^T A = pl_n, where I_n denotes the identity matrix of order n. Such a matrix is denoted by W(n, p). If squaring all its entries gives an incidence matrix of a SBIBD then W is called a balanced weighing matrix.

An Hadamard matrix, A = \{a_{ij}\}, is a W(n,n), that is, it is a square matrix of order n with entries a_{ij} \in \{1, -1\} which satisfies

AA^T = A^T A = \kappa I_n.

A generalized Hadamard matrix GH(gh,G) = (g_{ij}) = H over the group G of order g is a gh\times gh matrix such that

(i) g_{ij} \in G for all 1 \leq i, j \leq gh, and

(ii) \sum_{k=1}^{gh} g_{ik} g_{jk}^{-1} = \sum_{a \in G} a a^* whenever i \neq j where the summation is in the group ring R(G). We also write this as

HH^* = hG.

Suppose we have a matrix W with elements from an elementary abelian group G = \{h_1, h_2, \ldots, h_g\}, where W = h_1 A_1 + h_2 A_2 + \cdots + h_g A_g; here A_1, \ldots, A_g are

CONGRESSUS NUMERANTUM 59(1987), pp.265-274
v × b (0,1) matrices, and the Hadamard product $A_i^* A_j$ ($i \neq j$) is zero. Suppose 
$(a_1, \ldots, a_{ib})$ and $(b_{j1}, \ldots, b_{jb})$ are the $i$th and $j$th rows of $W$; then we define $WW^*$ by

$$(WW^*)_{ij} = (a_{i1}, \ldots, a_{ib})^T (b_{j1}^T, \ldots, b_{jb}^T)$$

with $\cdot$ designating the scalar product. Then $W$ is a generalized Bhaskar Rao design or GBRD if

(i) $WW^* = rl + \sum_{i=1}^{m} (c_i G) B_i$

(ii) $N = A_1 + \cdots + A_m$ satisfies $NN^T = rl + \sum_{i=1}^{m} \lambda_i B_i$,

that is, $N$ is the incidence matrix of a PBIBD $(m)$, and $(c_i G)$ gives the number of times a complete copy of the group $G$ occurs.

Such a matrix will be denoted by $GBRD_G(v, b, r, k, \lambda_1, \ldots, \lambda_m; c_1, \ldots, c_m)$. In this paper we shall only be concerned with $m = 1, c = \lambda_1 g$, and $B_1 = J - I$. In this case $N$ is the incidence matrix of a PBIBD $(1)$, that is a BIBD. Hence, the equations become:

(i) $WW^* = rl + \lambda G (J - I)$

(ii) $NN^T = (r - \lambda) J + \lambda J$.

Thus $W$ is a $GBRD_G(v, b, r, k, \lambda)$. Since $\lambda(v-1) = r(k-1)$ and $bk = vr$, we sometimes use the notation $GBRD(v, k, \lambda; G)$.

2. THE CONSTRUCTION

In his 1975 paper, Vanstone gave a powerful method for constructing BIBD from SBIBDs. We show his method applies to symmetric GBRD over groups which have no elements of order 2.

THEOREM 1. Suppose there is a symmetric GBRD$(v, k, \lambda; G)$, $|G|$ odd, then there is a GBRD$(v, k, \lambda; G)$.

Proof: We modify the construction Vanstone used to show that an SBIBD$(v, k, \lambda)$ yields a BIBD$(v, k, \lambda, \lambda; G)$.

Let $A = (a_{ij})$ be the incidence matrix of the GBRD$(v, k, \lambda; G)$. Label the columns of a $v \times \left[\begin{array}{c} \lambda \\ 2 \end{array}\right]$ matrix $B = (b_{ij})$, with the $n = \left[\begin{array}{c} v \\ 2 \end{array}\right]$ pairs from the set $\{1, \ldots, v\}$.
Consider the column labelled \(xy, (b_{1k}, \cdots, b_{nk})^T\), choose
\[b_{ik} = a_{ix}a_{iy}, \ i = 1, \ldots, v.\]

Clearly, every element of \(B\) is zero or a group element, as that was true of \(A\).

To establish the inner product property, we consider the inner product of two distinct rows
\[
\sum_{k=1}^{n} b_{ik}b_{jk}^{-1} = \sum_{1 \leq i < y \leq n} a_{ix}a_{iy}^{-1}a_{iy}^{-1}, \ i \neq j.
\]

We first note that, for any group \(G\) of order \(g\) with elements \(g_1, g_2, \cdots, g_v\)
\[G^2 = (g_1 + g_2 + \cdots + g_v)^2 = gG.\]

With \((G + \cdots + G)\) denoting \(t\) copies of \(G\)
\[(G + G + \cdots + G)^2 = tG^2 + 2\left[ \sum G^{2} \right] G^{2} = t^{2} gG.\]

Since \(g\) is odd and \(n = v = t_8\), if \(g_1, \cdots, g_v\) are the elements of a row of the GBRD, \(g_1^2, \ldots, g_v^2 = tG.

Hence, noting
\[(\sum x_j)^2 = \sum x_j^2 + 2 \sum x_i x_j,
\]
\[
\sum_{1 \leq i < j \leq n} a_{ix}a_{iy}^{-1}a_{iy}^{-1} = \frac{1}{2} \left[ \sum_{k=1}^{n} a_{ik}a_{jk}^{-1} \right]^2 - \frac{1}{2} \sum_{k=1}^{n} (a_{ik}a_{jk}^{-1})^2
\]
\[= \frac{1}{2}(G + G + \cdots + G)^2 - \frac{1}{2} g^2 G \quad (t \text{ copies})
\]
\[= \frac{1}{2}(t^2 g^{-1} G).
\]

Now, we know from Vanstone's result that a BIBD\((n,k,\lambda)\) gives a BIBD\((v, \lambda, \lambda, \lambda)\). Thus, we wish to show a GBRD\((v, k, \lambda; G)\) gives a BIBD\((n, \lambda, \lambda, \lambda)\).

Certainly, the underlying BIBD has these parameters. The GBRD\((v, k, \lambda; G)\) has \(t = \lambda / g\) copies of the group as the inner product of each pair of rows and the constructed GBRD needs to have \(\frac{t}{g}\) copies of the group as the inner product of each pair of rows. But
\[\frac{t}{g} = \frac{\lambda(\lambda-1)}{g} = \frac{\lambda(tg-1)}{g} = \frac{\lambda(tg-1)}{g},\]
as required.

Example 1. Let the group of order 3, \(Z_3\), have generator \(\omega\). Represent \(\omega\) by 1, \(\omega^2\) by 2 and \(\omega^3\) by 0. Then, the GH(6,\(Z_3\)) or GBRD(6, 6, 6;\(Z_3\)) is
yielding

\[
\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 \\
2 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 1 & 2 & 1 & 1 \\
1 & 2 & 2 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 2 & 1 & 1 \\
\end{array}
\]

a GBRD(6, 15, 15, 6, 15; \mathbb{Z}_3).

Example 2. Proceed as in Example 1, but represent the zero element by *. Then the GBRD(5, 4, 3; \mathbb{Z}_3)

\[
\begin{align*}
&\cdot & 0 & 0 & 0 & 0 \\
&0 & \cdot & 0 & 1 & 2 \\
&0 & 0 & \cdot & 2 & 1 \\
&0 & 1 & 2 & \cdot & 0 \\
&0 & 2 & 1 & 0 & \cdot 
\end{align*}
\]

yields the GBRD(5, 10, 6, 3, 3; \mathbb{Z}_3):

\[
\begin{array}{cccccccc}
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\
\cdot & 0 & 1 & 2 & \cdot & \cdot & \cdot & 1 & 2 & 0 \\
0 & \cdot & 2 & 1 & \cdot & 2 & 1 & \cdot & \cdot & 0 \\
1 & 2 & \cdot & 0 & 0 & \cdot & 1 & \cdot & 2 & \cdot \\
2 & 1 & 0 & \cdot & 0 & 2 & \cdot & 1 & \cdot & \cdot \\
\end{array}
\]

This method is so powerful when applied to generalized Hadamard matrices that we give it as a theorem in its own right.

3. USING GENERALIZED HADAMARD MATRICES IN THE CONSTRUCTION TO FORM BIBDS

**THEOREM 2.** Suppose there is a GH(tg; G), |G| = g odd. Then there is a GBRD( tg, \left[\frac{g}{2}\right], \left[\frac{g}{2}\right], G ). This can be used to form a

\[
GDD(\ g\ (g + 1), \left[\frac{g}{2}\right], \left[\frac{g}{2}\right], \ g + 1, \lambda_1 = 0, \lambda_2 = \frac{g}{2}(t g - 1), m = g, n = t g + 1).
\]

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COMMENT. The following construction is valid for any \( \text{GH}(2 \mid G; G) \) but these are presently only known for prime power orders \( |G| \). The BIBD's constructed would be multiples of biplanes \( \text{SBIBD}(2p^2 + p + 1, 2p + 1, 2) \) but these are not generally known as yet.

**Theorem 3.** Let \( p \) be any prime power. Then there exists a \( \text{BIBD}(2p^2 + p + 1, p(2p^2 + p + 1), p(2p + 1), 2p + 1, 2p) \).

**Proof.** We note a \( \text{GH}(2p, EA(p)) \) exists for every prime power (Jungnickle 1979), D.J. Street (1979)). Use Theorem 2 to form a \( \text{GBRD}(2p, p(2p-1), p(2p-1), 2p, p(2p-1); EA(p)) \). We replace each element of the GBRD by its \( p \times p \) permutation matrix representation to obtain a \( (0,1) \) matrix \( B \). Let \( e \) be the \( 1 \times p(2p-1) \) matrix of ones. Then

\[
A = \begin{bmatrix} I_p \times e \\ B \end{bmatrix}
\]

is a \( \text{GDD}(2p^2 + p, p^2(2p - 1), p(2p - 1), 2p + 1, 2p + 1, \lambda_1 = 0, \lambda_2 = (2p - 1)) \).

Now a \( \text{BIBD}(2p+1, p(2p+1), 2p, 2, 1) \) exists. Let \( C \) be the matrix obtained from this BIBD by replacing each 1 and 0 in its incidence matrix by the \( p \times 1 \) matrices of ones and zeros respectively. Then the matrix

\[
[C: A]
\]

has \( 2p^2 + p \) rows, \( 2p^2 + p^3 + p \) columns, \( 2p^2 + p \) ones per row, \( 2p \) or \( 2p + 1 \) ones per column and inner product \( 2p \). So if we let \( f \) be a \( 1 \times p(2p + 1) \) matrix of ones

\[
\begin{bmatrix} f & 0 \\ C & A \end{bmatrix}
\]

is a \( \text{BIBD}(2p^2 + p + 1, p(2p^2 + p + 1), p(2p + 1), 2p + 1, 2p) \).

\[ \square \]

**Corollary 4.** Let \( p \) be any prime power and \( q \) any integer. Then there exists a \( \text{PBIBD}(2p^2, p(p+q)(2p-1), (p+q)(2p-1), 2p, \lambda_1 = q(2p - 1), \lambda_2 = 2p - 1 + q) \).

**Proof:** As in the proof of Theorem 3, we use the \( \text{GH}(2p, EA(p)) \) to first form a \( \text{GBRD}(2p, p(2p-1), p(2p-1), 2p, \lambda_1 = q(2p - 1), \lambda_2 = 2p - 1 + q) \).

This then yields a \( \text{GDD}(2p^2, p^2(2p - 1), p(2p - 1), 2p, \lambda_1 = 0, \lambda_2 = 2p - 1) \),

A. Form \( C \) as before from a \( \text{BIBD}(2p, qp(2p-1), q(2p-1), 2, q) \).

Then \( [C: A] \) is a \( \text{PBIBD}(2p^2, p(p+q)(2p-1), (p+q)(2p-1), 2p, \lambda_1 = q(2p - 1), \lambda_2 = 2p - 1 + q) \).

\[ \square \]
Example 3. A $\text{GH}(6,E(3))$ exists so there is a $\text{GBRD}(6,10,6,10;E(3))$. This can be used with a $\text{BIBD}(7,21,6,2,1)$ to form a $\text{BIBD}(22,66,21,7,6)$.

Example 4. A $\text{GH}(18,E(9))$ exists, so there is a $\text{GBRD}(18,153,153,18,153;E(9))$. This is used with a $\text{BIBD}(19,171,18,2,1)$ to form a $\text{BIBD}(172,9 \cdot 172,171,19,18)$.

All the following constructions can be obtained by a similar, slightly modified technique.

**Theorem 5.** Suppose there exists a $\text{GH}(tg,G)$, $g = 1$ G | odd. Further suppose that there exists a $\text{BIBD}(tg + 1, s(tg + 1), t, t, \lambda)$. Then there exists a $\text{BIBD}(tg^2 + g + 1, s(tg^2 + g + 1), s(tg + 1), t + 1, \lambda\alpha t)$ where $s = \alpha g / (t-1)$ is an integer and $2\alpha g / (t-1) = \beta g / (t-1)$ for some $\alpha$ and $\beta$. In particular, if $\alpha = \lceil \frac{g}{2} \rceil$ and $\beta = tg - t + 1$, there is a

$$\text{BIBD}(tg^2 + g + 1, s(tg^2 + g + 1) \lceil \frac{g}{2} \rceil, s(tg + 1) \lceil \frac{g}{2} \rceil, tg + 1, s(tg + 1) \lceil \frac{g}{2} \rceil).$$

**Proof:** From theorem 1, there exists a $\text{GBRD}(tg, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil; G)$. We replace each element of $G$ by its $p \times p$ permutation matrix to form a $(0,1)$ matrix $E$. Further, let $e$ be the $(1, \lceil \frac{g}{2} \rceil$ matrix of ones. Then,

$$B = \begin{bmatrix} I_{g \times e} \\ E \end{bmatrix}$$

is a $\text{GDD}(g(tg + 1), g \lceil \frac{g}{2} \rceil, tg + 1, \lambda_1 = 0, \lambda_2 = \frac{1}{4} g(tg - 1), m = g, n = tg + 1)$. We now replace each 0 and 1 of the $\text{BIBD}(tg + 1, \lambda g(tg + 1) / (t - 1), \lambda g / (t - 1), t, \lambda)$ by the $g \times 1$ matrix of zeros and ones respectively to form a $\text{GDD}(tg+1,\lambda g(tg+1)/(t-1),\lambda g/(t-1),tg,\lambda_1 = \lambda g/(t-1),\lambda_2 = \lambda, m = g, n = tg+1)$, $A$.

We now form the following $(0,1)$ matrix:

$$C = \begin{bmatrix} 11 \cdots 11 & 00 \cdots 00 \\ \alpha \text{copies}A & \beta \text{copies}B \end{bmatrix}$$

The first row of $C$ has $\alpha \lambda g(tg + 1) / (t - 1)$ ones and has intersection $\alpha \lambda g / (t - 1)$ with the other rows of $C$.

Every other row of $C$ has $\alpha \lambda g / (t - 1) + \beta / 2 g(tg - 1)$ ones. So we require

$$\alpha \lambda g(tg + 1) / (t - 1) = \alpha \lambda g / (t - 1) + \beta / 2 g(tg - 1)$$

or

$$\alpha \lambda = \beta / 2 g(tg - 1)(t - 1) / (tg - t + 1) \tag{1}$$

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The intersection numbers for the rows are required to be equal, so we need
\[ \alpha \lambda g / (t - 1) = \alpha \lambda g / (t - 1) + \beta \cdot 0 = \alpha \lambda + \beta \lambda g / (t-1). \]

or, as in (1)
\[ \alpha \lambda = \beta \lambda g / (t-1) / (tg - t + 1). \]

Thus C is a BIBD\((tg^2 + g + 1, \alpha \lambda g (tg^2 + g + 1)/(t-1), \alpha \lambda g (tg + 1)/ (t-1), \alpha \lambda g / (t-1)).\)

Where \( \lambda g / (t - 1) = s \) an integer, and a possible solution for \( \alpha \) and \( \beta \) is
\[ \alpha = s \left( g^2 + g + 1 \right) \left( \frac{tg}{2} \right), \beta = s (tg - t + 1). \]

That is, C is a BIBD\((tg^2 + g + 1, \alpha \lambda g (tg^2 + g + 1)/(t-1), \alpha \lambda g (tg + 1)/ (t-1), \alpha \lambda g / (t-1)).\)

COROLLARY 6. Let \( g \) and \( g-1 \) be prime powers, \( g \) odd. If there exists a \( BIBD(g^2 - g + 1, g (g^2 - g + 1), g (g - 1), g - 1, g, g - 2) \) then there exists a \( BIBD(g^2 - g + 1, \alpha \lambda g (g^2 - g + 1), g (g^2 - g + 1), \alpha \lambda g (g^2 - g + 1), g^2 - g + 1, \alpha \lambda g (g - 1)), \)

where \( 2 \alpha (g^2 - g + 2) = \beta (g - 1)(g^2 - g - 1) \) has an integer solution.

Proof: By a Theorem of Rajkundila (1978) and Seberry (1981), a \( GH(g(g-1); EA(g)) \) always exists in these cases.

Remark. The \( BIBD \) obtained would be a multiple of an \( SBIBD(g^3 - g^2 + g + 1, g^2 - g + 1, g - 1) \) which theoretically, can never exist, as \( g^3 - g^2 + g + 1 \) is even and \( k - \lambda = g^2 - 2g + 2 \) is not a square.

Example 5. Let \( g = 5 \). There exists a \( BIBD(21, 105, 20, 4, 3) \). Hence, there exists a \( BIBD(106, 38, 5, 106, 38, 5, 21, 38, 5, 4), \alpha = 38 \). This is a multiple of the \( SBIBD(106, 21, 4) \) which is non-existent.

COROLLARY 7. Let \( g \) be an odd prime power. Let \( \alpha = 2(4g - 1) \). Then there is a \( BIBD(4g^2 + g + 1, 2g^2 + (4g^2 + g + 1)(4g - 1), 2g^2 (4g + 1)(4g - 1), 4g + 1, 8g^2 (4g - 1)). \)

Proof: Dawson (1985) has shown a \( GH(4g, 4g; EA(g)) \) always exists. Also, the required \( BIBD(g + 1, g (4g + 1), 4g, 4, 3) \) always exists and so, with \( \alpha = 2(4g - 1), \beta = 4g - 3 \) in Theorem 5, we get the result.

Remark. This would be a multiple of the \( SBIBD(4g^2 + g + 1, 4g + 1, 4) \) but this can only exist (since \( 4g^2 + g + 1 \) is even) if \( k - \lambda = 4g - 3 \) is a square.

Example 6. Let \( g = 9 \). Then \( \alpha = 70 \) and a \( BIBD(334, 70, 9, 334, 70, 9, 37, 37, 36, 70) \) exists.

COROLLARY 8. Let \( g = 3^h \). Then there exists
\( BIBD(4g^2 + g + 1, \alpha \lambda g (4g^2 + g + 1)/3, \alpha \lambda g (4g + 1)/3, 4g + 1, 4\alpha \lambda g/3) \)

where \( 2 \alpha \lambda (4g - 3) = 12 \beta (4g - 1) \) for some \( \alpha \) and \( \beta \). In particular, if \( \alpha \lambda = 2(4g - 1) \)
and $\beta = (4g - 3)/3$, there is a
\[
\text{BIBD}(4g^2 + g + 1, 2g(4g - 1)/(4g^2 + g + 1), 6, 2g, 4g + 1, 6, 4g + 1, 8g(4g - 1)/3).
\]

Proof: We again use the GH(4g, EA(g)) found by Dawson (1985). We note that a
BIBD(4g + 1, $\lambda g(4g + 1)/3, 4A_g/3, \lambda$) exists for all $\lambda$. We use these in Theorem 5
to get the result.

Remark. The constructed designs are also multiples of an SBBBD(4g^2 + g + 1, 4g + 1, 4) which never exists as $4g^2 + g + 1$ is even and $k - \lambda = 4g - 3$ is never a square for $g = 3^h, h > 1$.

**COROLLARY 9.** Let $p$ be an odd prime power. Suppose there exists a
BIBD($p^i + 1, q p^j (p^i + 1), q p^j, p^{i - j}, q (p^{i-j} - 1)$) where $i \geq j$, and $q$ are integers.
Then there exists a
\[
\text{BIBD}(p^{i+j} + p^{i-j} + 1, q p^{i-j}(p^{i+j} + 1), q p^{i-j}, p^{i-j} + 1, q p^{i-j})
\]
where $2q(p^{i-j} - p^{i-j} + 1) = \beta p^{i-j}(p^{i-j} - 1)$, there is a
\[
\text{BIBD}(p^{i+j} + p^{i-j} + 1, p^{i-j}(p^{i-j} - 1)(p^{i-j} + 1), p^{i-j}(p^{i-j} + 1), p^{i-j} + 1, p^{2i-j}(p^{i-j} - 1)).
\]

Proof: Use the GH($p^i$, EA($p^i$), $i > j$ given by Drake (1979) or Butson (1963).

Remark. This would be a multiple of the SBBBD($p^{i+j} + p^{i-j}, p^{i-j}, p^{i-j}$). Since
$p^{i+j} + p^{i-j}$ is odd, in order for this to exist, the diophantine equation
\[
z^2 = (p^i - p^{i-j} + 1)x^2 + (-1)^{2(i-j)}y^2
\]
must have a solution in the integers for $x, y, z$ not all zero.

**Example 7.** Let $i = 2, j = 1, q = 1$ and $p = 5$. A BIBD(26, 130, 25, 5, 4) exists.
Hence a BIBD(131, 600-131, 600, 26, 26, 600-5) exists.

4. USING GENERALIZED WEIGHING MATRICES IN THE CONSTRUCTION

As noted in Seberry (1979), and Geramita and Seberry (1979), infinite families of
GW matrices are known.

**THEOREM 10.** Let $p^r$ be a prime power and $q | p^r - 1$, $q$ odd. Then there exists a
\[
\text{GBBD}(p^r + 1, q p^r(p^r + 1), q p^r(p^r - 1), p^r - 1, q p^r(p^r - 2), Z_{q})
\]
and $B$, a GDD with parameters
\[
(q(p^r + 1), q p^r(p^r + 1), q p^r(p^r - 1), p^r - 1, q p^r(p^r - 2), Z_{q}, m = q, n = p^r + 1).
\]
Hence if, $q | p^r - 1$, there exists a
\[
\text{BIBD}(p^{2r} - 1, q p^{2r} - 2(p^r + 1), q p^{2r} - 2, p^r - 1, q(p^r - 2)).
\]
If $q | p^r - 1$, $q \neq p^r - 1$ and there exists a BIBD($p^r + 1, b$, $p$, $(p^r - 1)/q$, $\lambda$), $A$,
\[ \lambda q^{p'} = p(p^r - q - 1). \] Using \( A \) to form \( a \)

\[ \text{GDD}(q(p^r + 1), q, p, p^r - 1, \lambda_1 = r, \lambda_2 = \lambda) \]

then

\[ [\alpha \text{ copies of } A : \beta \text{ copies of } B] \]

where \( 2q\alpha(p - \lambda) = \beta(p^r - 1)(p^r - 2) \) gives a BIBD \((q(p^r + 1), B, R, p^r - 1, \alpha p)\).

**Proof:** We note first that a GW\((p^r+1, p^r, p^r-1; Z_q)\) exists for all \( p^r \). The proof, then, is identical to the first part of the proof of Theorem 3.

**Example 8.** A GW\((17, 16, 15; Z_4), q = 15, 5 \) and \( 3 \) exists. This gives a BIBD\((15\cdot17, 127\cdot17, 127, 15, 7)\). Also, we have GDD\((5\cdot17, 40\cdot17, 120, 15, \lambda_1 = 0, \lambda_2 = 21, m = 5, n = 17)\) and a GDD\((3\cdot17, 24\cdot17, 120, 15, \lambda_1 = 0, \lambda_2 = 35, m = 3, n = 17)\). Since BIBD\((17, 8:17, 24, 3, 3)\) and BIBD\((17, 4:17, 20, 5, 5)\) exist, we have a BIBD\((85, 34:24, 6:24, 15, 24)\) with \( q = 5, r = 24, \lambda = 3, \alpha = \beta = 1 \) and a BIBD\((51, 1700, 500, 15, 140)\) with \( q = 3, r = 20, \lambda = 5, \alpha = 7, \) and \( \beta = 3 \).

**Example 9.**

We note that there exists a GW\((p^{n+1} - 1)(p - 1); Z_q)\) for all \( q \mid p^n(p - 1) \). So we can choose \( q \) odd and proceed as in the previous theorem. We do not give full results but note some examples: the GW\((21, 16; Z_3)\) gives a GBRD\((21, 12, 66; Z_3)\) and a BIBD\((63, 12, 22:60)\), the GW\((31, 25; Z_2)\) gives a GBRD\((31, 20, 190; Z_2)\) and a BIBD\((155, 20, 19:20)\), and the GW\((85, 64; Z_3)\) gives a GBRD\((85, 48, 24;47; Z_3)\) and a BIBD\((255, 24, 94:336)\).

5. REFERENCES


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