A Construction for Orthogonal Designs with Three Variables

Jennifer Seberry
Department of Computer Science
University College
The University of New South Wales
Australian Defence Forces Academy
Canberra, A.C.T. 2600
AUSTRALIA

TO ALEX ROSA ON HIS FIFTIETH BIRTHDAY

ABSTRACT

We show how orthogonal designs $OD(48p^2t;16p^2t,16p^2t)$ can be constructed from an Hadamard matrix of order $4p$ and an $OD(4t;t,t,t)$. This allows us to assert that $OD(48p^2t;16p^2t,16p^2t)$ exist for all $t,p \leq 102$ except possibly for $t \in \{67,71,73,77,79,83,86,89,91,97\}$. These designs are new.

1. Introduction

Let $H = (h_{ij})$ be a matrix of order $h$ with $h_{ij} \in \{1,-1\}$. $H$ is called an Hadamard matrix of order $n$, if $HH^T = hI_n$ where $I_n$ denotes the identity matrix order of $h$.

An orthogonal design $A$, of order $n$, type $(p_1,p_2,\ldots,p_n)$, denoted $OD(n;p_1,p_2,\ldots,p_n)$, on the commuting variables $(\pm x_1,\pm x_2,\ldots,\pm x_n,0)$ is a square matrix of order $n$ with entries $\pm x_k$ where each $x_k$ occur $p_k$ times in each row and column such that the rows are pairwise orthogonal.

In other words

$$AA^T = (p_1x_1^2 + \cdots + p_nx_n^2)I_n.$$  

It is known that the maximum number of variables is an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^b$, $b$ odd, set $a = 4c + d$, $0 \leq d < 4$, then $\rho(n) = 8c + 2^d$.

$OD(4t;t,t,t,t)$, otherwise called Baumert-Hall arrays, and $OD(2^a;a,2^b-a-b)$ have been extensively used to construct Hadamard matrices and weighing matrices. For details see Geramita and Seberry (1976).
Geramita, Geramita and Wallis (Seberry) observed (see Geramita and Seberry [1979, §4.3]) that if \( A, B, C, D \) are four circulant or type 1 matrices of order \( n \) satisfying

\[
AA^T + BB^T + CC^T + DD^T = (\sum_{i=1}^{n} p_i x_i^2) I_n
\]

then \( A, B, C, D \) can be used in the Goethals-Seidel array or (J. Seberry Wallis-Whiteman array)

\[
\begin{bmatrix}
A & BR & CR & -DR \\
-BR & A & D^T R & -C^T R \\
CR & -D^T R & A & B^T R \\
DR & C^T R & -B^T R & A
\end{bmatrix}
\]

(1).

to form an orthogonal design \( OD(4n; p_1, p_2, \ldots, p_n) \).

2. Background

Kharaghani (1988) defined \( C_{ij} = [h_{ij}, h_{ji}] \) and applying that to Hadamard matrices of order \( 4p \), obtained matrices satisfying

\[
C_i C_j = 0, \ i \neq j \tag{2}
\]

\[
\sum_{i=1}^{4p} C_i^2 = (4p)^2 I_{4p}.
\]

He then used this to show there are Bush-type (blocks \( J_{4p} \) down the diagonal) and Seifert-type \( (h_{ij} = -1 \Rightarrow h_{ji} = 1 \) and not necessarily vice versa) Hadamard matrices. By using a symmetric Latin square he could also have shown that regular symmetric Hadamard matrices with constant diagonal of order \((4p)^2\) could be constructed by his method.

Hammer, Sarvate and Seberry applied Kharaghani's method to \( OD(n; a_1, \ldots, a_n) \) and in particular \( OD(4t; t, t, t, t) \) and \( OD(4as; a, a, a) \) obtaining existence of \( OD(48s^2, 12s^2, 12s^2) \) and \( OD(80s^2, 20s^2, 20s^2) \) from \( OD(48s^2, 4s^2, 4s^2) \) and \( OD(4t, t, t, t) \). Seberry (to appear) extended this further to obtain \( OD(16ks^2, 4ks^2, 4ks^2, 4ks^2) \) for \{1,3,5,\ldots\}.

We modify their techniques to obtain new orthogonal designs.

3. Construction

Let \( C_1, C_2, \ldots, C_{4p} \) be the Kharaghani matrices of order \( 4p \) obtained from an orthogonal design \( OD(4p; p, p, p, p) \).

Let \( \alpha, \beta, \gamma \) be commuting variables and write

\[ [\alpha C_1 : \alpha C_2 : \ldots : \alpha C_{4p} : \beta C_{p+1} : \ldots : \beta C_{2p} : \gamma C_{3p+1} : \ldots : \gamma C_{4p}] \]

\[ [\alpha C_{3p+1} : \ldots : \alpha C_{4p} : \gamma C_{p+1} : \ldots : \gamma C_{2p} : \beta C_{2p+1} : \ldots : \beta C_{4p}] \]
for the first rows of two block circulant matrices $W_1$ and $W_2$. It can be checked that $W_1 W_2^T = W_2 W_1^T$. Write

\[
\begin{align*}
& [aC_0 : \ldots : aC_{p-1}] \quad [bC_0 : \ldots : bC_{p-1}] \\
& [aC_p : \ldots : aC_{2p-1}] \quad [bC_p : \ldots : bC_{2p-1}]
\end{align*}
\]

for the first rows of two block back-circulant matrices $W_3$ and $W_4$. It can be checked that $W_3 W_4^T = W_4 W_3^T$.

Now by virtue of being circulant and back-circulant

$$W_i W_j^T = W_j W_i^T \quad i \in \{1, 2\}, \quad j \in \{3, 4\}.$$ 

Example. Let $p = 3$ so there are 12 matrices of order 12, $C_1, \ldots, C_{12}$. Then

$$W_1 = 
\begin{bmatrix}
C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 \\
C_0 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 \\
C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_0 \\
C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_0 & C_1 \\
C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_0 & C_1 & C_2 \\
C_5 & C_6 & C_7 & C_8 & C_9 & C_0 & C_1 & C_2 & C_3 \\
C_6 & C_7 & C_8 & C_9 & C_0 & C_1 & C_2 & C_3 & C_4 \\
C_7 & C_8 & C_9 & C_0 & C_1 & C_2 & C_3 & C_4 & C_5 \\
C_8 & C_9 & C_0 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\
C_9 & C_0 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \\
C_0 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 \\
C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 \\
C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_0
\end{bmatrix}
$$

$$W_2 = 
\begin{bmatrix}
C_{10} & C_{11} & C_{12} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 \\
C_{11} & C_{12} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} \\
C_{12} & C_{11} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} \\
C_{10} & C_{11} & C_{12} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} \\
C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_0 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_0
\end{bmatrix}
$$

but $C_1 C_1^T$ is symmetric and so $W_1 W_2^T = W_2 W_1^T$.

$W_3$ and $W_4$ are formed similarly but are back-circulant in blocks (i.e., type 2) in the language of Seberry-Wallis and Whitman).

Thus $W_1, W_2, W_3, W_4$ are Williamson-type matrices of order $12p^2$. They can be used to replace the variables of an $OD(4t; 1, 1, 1)$ to get an $OD(4p^2; 16p^2, 16p^2, 16p^2)$. In general they can be used to replace the variables of an $OD(4t; t, t, t)$ so we have
Theorem: If there is an Hadamard matrix of order 4p and an OD\((4t; t, t, t)\) then there is an OD\((48p^2t; 16p^2t, 16p^2t, 16p^2t)\).

Since Hadamard matrices of order 4p exist for all \(p \leq 102\) and OD\((4t; t, t, t)\) exist for all \(t \leq 102\) except possibly for \(t \in S, S = \{67, 71, 73, 77, 79, 83, 86, 89, 91, 97\}\) (see Seberry (1986a)) we have

Corollary: OD\((48p^2t; 16p^2t, 16p^2t, 16p^2t)\) exist for all \(t, p \leq 102\) except possibly for \(t \in S\).

Acknowledgement

The author's research is supported in part by grants from ACRB and AERB.

References