GENERALIZED BHASKAR RAO DESIGNS OF BLOCK SIZE THREE

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Abstract: We show that the necessary conditions

\[ \lambda \equiv 0 \pmod{|G|}, \]
\[ \lambda(v - 1) \equiv 0 \pmod{2}, \]
\[ \lambda v(v - 1) \equiv \begin{cases} 0 \pmod{6} & \text{for } |G| \text{ odd}, \\ 0 \pmod{24} & \text{for } |G| \text{ even}, \end{cases} \]

are sufficient for the existence of a generalized Bhaskar Rao design \( GBRD(v, b, r, 3, \lambda; G) \) for the elementary abelian group \( G \), of each order \( |G| \).

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1. Introduction

For the definitions and notations used in this paper we refer the reader to Lam and Seberry (1984). Since the underlying structure for a \( GBRD(v, 3, \lambda; G) \) is a \( BIBD(v, 3, \lambda) \) we have the following necessary conditions for existence:

\[ \lambda \equiv 0 \pmod{|G|}, \]  \hfill (1.1)
\[ \lambda(v - 1) \equiv 0 \pmod{2}, \]  \hfill (1.2)
\[ \lambda v(v - 1) \equiv 0 \pmod{6}. \]  \hfill (1.3)

In Lam and Seberry (1984) the extra condition

\[ |G| \equiv 0 \pmod{2} \Rightarrow b \equiv 0 \pmod{4} \]  \hfill (1.4)

was established.

We also use the notation \( EA(H) = EA(\prod_{i=1}^{s} p_i^{e_i}) \) for the elementary abelian group \( Z_{p_1} \times Z_{p_2} \times \cdots \times Z_{p_s} \times \cdots \times Z_{p_s} \times \cdots \times Z_{p_s} \) where \( Z_{p_i} \) occurs \( r_i \) times with \( \prod_{i=1}^{s} p_i^{e_i} \) the prime decomposition of the order of the group \( H \).
In this paper we establish that the necessary conditions are sufficient for the existence of a GBRD($n, 3, \lambda; \text{EA}(H))$.

Lam and Seberry (1984), building on the results of Seberry (1982, 1984), proved:

**Theorem 1.1.** The necessary conditions (1.1)–(1.4) are sufficient for the existence of a GBRD($n, 3, \lambda; G$) when

(i) $|G|$ is odd,

(ii) $G = Z_3^*$,

(iii) $G = Z_3^r \times H$ where $3 \nmid |H|$ and $r \geq 1$.

In this paper we establish existence for the remaining group orders.

2. Existence of GBRD on $Z_3 \times Z_3$

**Theorem 2.1.** The necessary conditions

$$\lambda \equiv 0 \pmod{6},$$

$$\lambda \nu (\nu - 1) \equiv 0 \pmod{24}$$

are sufficient for the existence of a GBRD($n, 3, \lambda; Z_3 \times Z_3$).

**Proof.** From Section 1 these are clearly necessary conditions: in the case $\lambda \equiv 6 \pmod{12}$ they become $\nu (\nu - 1) \equiv 0 \pmod{4}$ and for $\lambda \equiv 0 \pmod{12}$ there is no condition on $\nu$. Hence we consider these cases separately.

**Case 1:** $\lambda = 6$. The necessary condition becomes $\nu (\nu - 1) \equiv 0 \pmod{4}$. We first establish the existence of $\text{GBRD}(4, 3, 6; Z_3 \times Z_3)$ for $\nu \in \mathcal{K}_4^3 = \{4, 5, 8, 9, 12\}$.

Using 0, 1, 2, 3, 4, 5 for the elements, additively, of $Z_6$ we have that

$$D_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 4 & 5 & \ldots & \ldots & 0 & 0 & 0 \\
0 & 2 & 1 & \ldots & 3 & 4 & 5 & 3 & 2 & 4 \\
\ldots & \ldots & 3 & 2 & 1 & 5 & 4 & 0 & 1 & 5 & 3
\end{pmatrix}$$

is a GBRD(4, 3, 6; $Z_6$). The initial blocks developed as

- $D_5 = \{(0, 0, 1, 2, 1), (0, 0, 1, 2, 2), (0, 0, 1, 4, 3, 2), (0, 0, 1, 5, 3, 2)\}$ (mod 5, $Z_6$),
- $D_8 = \{(0, 0, 1, 2, 1), (0, 0, 1, 2, 2), (0, 0, 1, 4, 3, 2), (0, 0, 1, 5, 3, 2), (0, 0, 2, 4, 1, 4), (\infty, 1, 4, 2), (\infty, 1, 4, 2)\}$ (mod 7, $Z_6$),
- $D_{12} = \{(0, 0, 1, 1, 0, 1), (0, 0, 1, 2, 9, 3), (0, 0, 1, 3, 8, 3), (0, 0, 1, 4, 1, 3), (0, 0, 1, 10, 5, 3), (0, 0, 4, 5, 4), (0, 0, 2, 5, 0), (0, 0, 3, 0, 4, 3), (0, 0, 3, 2, 3, 3), (\infty, 0, 2, 2), (\infty, 0, 2, 2)\}$ (mod 11, $Z_6$),
- $D_{12} = \{(0, 0, 1, 1, 0, 1), (0, 0, 1, 2, 9, 3), (0, 0, 1, 3, 8, 3), (0, 0, 1, 4, 1, 3), (0, 0, 1, 10, 5, 3), (0, 0, 4, 5, 4), (0, 0, 2, 5, 0), (0, 0, 3, 0, 4, 3), (0, 0, 3, 2, 3, 3), (\infty, 0, 2, 2), (\infty, 0, 2, 2)\}$ (mod 11, $Z_6$),
- $D_{12} = \{(0, 0, 1, 1, 0, 1), (0, 0, 1, 2, 9, 3), (0, 0, 1, 3, 8, 3), (0, 0, 1, 4, 1, 3), (0, 0, 1, 10, 5, 3), (0, 0, 4, 5, 4), (0, 0, 2, 5, 0), (0, 0, 3, 0, 4, 3), (0, 0, 3, 2, 3, 3), (\infty, 0, 2, 2), (\infty, 0, 2, 2)\}$ (mod 11, $Z_6$),
give the required designs for \( u = 5, 8, 12 \). A GBRD\((9, 3, 6; \mathbb{Z}_6)\) with a subdesign on 4 points is

\[
D_4 = \begin{bmatrix}
e e e e e & e e e e & e e e e \\
x s & x s & x s & x s \\
y s & y s & x s & x s \\
y s & y s & y s & i x \\
i x & i x & i y & y t \\
y t & y t & y t & y t \\
\end{bmatrix}
\]

where

\[
e = (0, 0, 0),
\]
\[
x = (0, 1, 2),
\]
\[
s = (3, 4, 5),
\]
\[
y = (0, 2, 1),
\]
\[
l = (3, 5, 4),
\]
so that

\[
x \cdot y^{-1} = (0, 5, 1),
\]
\[
x \cdot l^{-1} = (3, 2, 4),
\]
\[
s \cdot l^{-1} = (0, 5, 1),
\]
\[
s \cdot y^{-1} = (3, 2, 4).
\]

Now using Theorem 2.2 of Lam and Seberry (1984) with Hanani's theorem (Hall (1967), Lemma 15.5.1) we only need to establish the existence of GBRD\((u, 3, 6; \mathbb{Z}_6)\) for \( u \in \mathbb{K}_4^+ = \{4, 5, 8, 9, 12\} \) to establish the existence of all GBRD\((u, 3, 6; \mathbb{Z}_6)\) for \( u = 0 \) or \( 1 \) (mod 4), \( u \geq 4 \). Thus we have all these designs and by taking \( t \) copies (\( t \) odd) we have GBRD\((u, 3, 6; \mathbb{Z}_6)\) for all \( \lambda = 6 \) (mod 12).

**Case 2:** \( \lambda = 12 \). There are now no necessary conditions on \( u \). We observe that if \( u = 2p + 1, \ p \) a positive integer, then GBRD\((2p + 1, 3, 12; \mathbb{Z}_6)\) are obtained by developing the blocks (modulo \( 2p + 1, \mathbb{Z}_6)\)

\[
(0_0, i_1, 2p + 1 - i_1), \ i = 1, \ldots, 2p,
\]
\[
(0_0, j_1, 2p + 1 - j_1), (0_0, j_2, 2p + 1 - j_2), \ j = 1, \ldots, p.
\]

For \( u = 2p + 2, \ p \geq 2 \) integer, a GBRD\((2p + 2, 3, 12; \mathbb{Z}_6)\) can be obtained by developing the following blocks (modulo \( 2p + 1, \mathbb{Z}_6)\):

\[
(0_0, i_1, 2p + 1 - i_1), \ i = 3, \ldots, 2p,
\]
\[
(0_0, j_1, 2p + 1 - j_1), (0_0, j_2, 2p + 1 - j_2), \ j = 1, \ldots, p,
\]
\[
(\infty_0, 0_1, 2p - 1_1), (\infty_0, 0_1, 2p - 3_1), (\infty_0, 0_2, 2p_0),
\]
\[
(\infty_0, 0_1, 2p - 1_1), (\infty_0, 0_2, 1_1), (\infty_0, 0_4, 2_1).
\]

The case for \( u = 4 \) is obtained by taking two copies of the GBRD\((4, 3, 6; \mathbb{Z}_6)\) given in case 1 above.
Hence we have constructed a $\text{GBRD}(u, 3, 12; Z_6)$ for every $u$. Taking $t$ copies gives us $\text{GBRD}(u, 3, 12t; Z_6)$ for every $\lambda = 12t = 0 \pmod{12}$. Hence we have the theorem.

3. Existence of $\text{GBRD}$ on $Z_2 \times Z_2 \times Z_3$

**Theorem 3.1.** The necessary condition $\lambda = 0 \pmod{12}$ is sufficient for the existence of a $\text{GBRD}(u, 2, 12t; EA(12))$.

**Proof.** For the group $G = EA(12)$ the only necessary condition for the existence of a $\text{GBRD}_{2t}(u, 3, \lambda)$ is that $\lambda = 0 \pmod{12}$.

To establish existence we use Theorem 2.2 of Lam and Seberry (1984) with Hanani’s theorem (Hall (1967), Lemma 15.5.2). Thus we must establish the existence of $\text{GBRD}(u, 3, 12; EA(12))$ for $v \in K_3 = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 22\}$ to establish existence for all $u \equiv 4$.

From Lam and Seberry (1984) we see that $\text{GBRD}(u, 3, 4; Z_2 \times Z_2)$, say $A$, exists for $v = 0, 1 \pmod{3}$. Writing $W = (w_{ij})$ for

$$W = \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix},$$

the generalized Hadamard matrix $GH(3, Z_3)$, we have, replacing every $A_{ij}$ of $A$ by the vector of ordered pairs $[(A_{ij}, w_{ij})]$, the required design. Hence we have the existence of $\text{GBRD}(u, 3, 12; Z_2 \times Z_2 \times Z_3)$ for $v = 4, 6, 7, 9, 10, 12, 15, 18, 19, 22$.

Similarly, from Seberry (1982), we have a $\text{GBRD}(u, 3, 3; Z_3)$ for all odd $u$. Thus, combining these designs with three rows of

$$H = \begin{pmatrix}
e & e & e & e \\
e & a & b & ab \\
e & b & ab & a \\
e & ab & a & b
\end{pmatrix}, \quad a^2 = b^2 = e, \quad ab = ba,$$

the generalized Hadamard matrix $GH(4, Z_2 \times Z_2)$, we have the existence of $\text{GBRD}(u, 3, 12, EA(12))$ for all odd $v$ and in particular for $v = 5, 11$ and 23.

A $\text{GBRD}(8, 3, 12; EA(12))$ is constructed by the use of Theorem 2.2 with the $\text{GBRD}(8, 4, 3; Z_4)$ obtained by developing the initial blocks $(\infty, 1, 2_{4w}, 4_{4w})$ and $(0, 1_{4w}, 2_{4w}, 4_{4w}) \pmod{7, Z_3}$, and the $\text{GBRD}(4, 3, 4; Z_2 \times Z_2)$

$$\begin{pmatrix}
e & e & e & e & e & e & e & e \\
e & a & b & ab & a & b & a & b \\
e & b & a & ab & b & a & b & ab \\
e & ab & a & b & ab & a & b & ab
\end{pmatrix}, \quad a^2 = b^2 = e, \quad ab = ba.$$
A GBRD(14,3,12; EA(12)) is obtained by developing the following initial blocks (modulo 13, EA(12)):

\[ (0_1, i_{n}, 13 - i_{abc}), (0_1, i_{n}, 13 - i_{abc}), (0_1, i_{n}, 13 - i_{xy}), \quad i = 1, \ldots, 6, \]
\[ (0_1, i_{n}, 13 - i_{xy}), \quad i = 3, 4, 5, 6, \]
\[ (\infty_1, 0_1, 12 - \omega), \quad (\infty_1, 0_1, 12 - \omega), \quad (\infty_1, 2_1, 12 - \omega), \quad (\infty_1, 0_1, 2_1), \quad (\infty_1, 0_1, 1_1), \]

where \( a^2 = b^2 = \omega^3 = 1 \) and all elements commute.

Thus a GBRD(14,3,12; EA(12)) exists for all \( v \in K_3^2 \) and hence, for all \( v \). GBRD for \( \lambda = 12t \) are obtained by taking \( t \) copies of the design for \( \lambda = 12 \), giving the result. □

4. Existence of GBRD on \( Z_2 \times Z_2 \times Z_2 \times Z_3 \)

**Theorem 4.1.** The necessary condition \( \lambda = 0 \) (mod 24), is sufficient for the existence of GBRD(14,3,12; EA(24)).

**Proof.** For the group EA(24) the only necessary condition is \( \lambda = 0 \) (mod 24). Hence, as in the previous section, it is only necessary to establish the existence of GBRD(14,3,12; EA(24)) for \( v \in K_2^4 \).

From Seberry (1982) a GBRD(u, 3, 3; Z_2) exists for all odd \( u \) and so we use these designs with three rows of the generalized Hadamard matrix \( GH(8, Z_2^2) = GBRD(14,3,12; Z_2 \times Z_2 \times Z_2) \), say

\[
\begin{pmatrix}
  e & e & e & e & e & e & e & e \\
  e & a & b & a & b & c & a & c \\
  a & b & c & a & b & c & a & b \\
\end{pmatrix}, \quad a^2 = b^2 = e^2 = e, \quad (4.1)
\]

where the elements commute, in Theorem 2.2 of Lam and Seberry (1984) to obtain GBRD(u, 3, 24; EA(24)) for all odd \( u \).

From Theorem 2.1 a GBRD(u, 3, 6; Z_3) exists for \( u = 0, 1, 4, 5, 8, 9 \) (mod 12) and so may be used with three rows of (4.2), the \( GH(4, Z_3^2) = GBRD(3,3,4; Z_2 \times Z_3) \) in Theorem 2.2 of Lam and Seberry (1984) to obtain GBRD(u, 3, 24; EA(24)) for these \( u \) and in particular for \( u \in \{4, 8, 12 \} \). From Lam and Seberry (1984) we see a GBRD(u, 3, 8; Z_2 \times Z_2 \times Z_2) exists for \( u(u-1) = 0 \) (mod 3): these designs may be used with (3.1), the \( GH(3, Z_3) = GBRD(3,3,3; Z_3) \) in Theorem 2.2 of Lam and Seberry (1984) to obtain GBRD(u, 3, 24) for these \( u \) including \( u \in \{6, 10, 18, 22 \} \).

There exists a GBRD(14,3,24; EA(24)). It is obtained by developing the following blocks modulo 13:

\[ (0_1, i_{n}, 13 - i_{xy}), (0_1, i_{n}, 13 - i_{abc}), (0_1, i_{n}, 13 - i_{abc}), (0_1, i_{n}, 13 - i_{abc}), \]
\[ (0_1, i_{n}, 13 - i_{abc}), (0_1, i_{n}, 13 - i_{abc}), (0_1, i_{n}, 13 - i_{abc}), \]
\[ (0_1, i_{n}, 13 - i_{abc}), \quad i = 1, 2, 3, 4, 5, 6. \]
A GBRD(14, 3, 24; EA(24)) can be obtained by developing the same blocks modulo 13 except that the last initial block, \((0_1, l_{cu}, 13 - l_{abu})\), \(i = 1, 2, \ldots, 6\), should be replaced by

\[
(0_1, 5_{cu}, 8_{abu}), (0_1, 1_{cu}, 7_{abu}), \\
(\infty, 0_{bcu}, 2_{bcu}), (0_1, 0_{bcu}, 3_{cu}), (\infty, 0_{bcu}, 4_{bcu}), \\
(\infty, 0_{bcu}, 5_{bcu}), (0_1, 1_{bcu}, 6_{bcu}), (\infty, 0_{bcu}, 7_{bcu}), (\infty, 0_{bcu}, 8_{bcu}),
\]

which should also be developed modulo 13.

Hence a GBRD(6, 3, 24; EA(24)) exists for all \(d \in K^2\) and hence, as in the proof of Theorem 3.1 for all \(v\). GBRD for \(\lambda = 24t\) are obtained by taking \(t\) copies of the design for \(\lambda = 12\) giving the result. \(\Box\)

5. Existence of Bhaskar Rao designs with block size three

**Theorem 5.1.** The necessary conditions (1.1)-(1.4) are sufficient for the existence of a generalized Bhaskar Rao design GBRD(6, 3, \(\lambda\); G) on the elementary abelian group G for every order \(|G|\).

**Proof.** From Theorem 1.1 it is merely necessary to consider the cases \(G = Z_2^r \times H\), \(r \geq 1\) where \(3 \mid |H|\). We have three basic cases:

(i) \(|G| = 6p\), \(p\) odd, with necessary condition \(pv(v - 1) \equiv 0 \pmod{4}\),

(ii) \(|G| = 12p\), \(p\) odd, with no condition on \(v\),

(iii) \(|G| = 24p\), \(p\) odd, with no condition on \(v\).

We use Theorems 2.1, 3.1 and 4.1 respectively to obtain the design for \(|G| = 6\), \(12\) and \(24\) and makes repeated use of theorem 2.2 of Lam and Seberry (1984) to obtain the result. For orders \(2^sp\), \(s \geq 4\), use \(|G| = 12\) or \(24\) and repeated use of GBRD(3, 3, 4; \(Z_2 \times Z_2\)) according as \(s\) is even or odd. \(\Box\)

6. Application

Proceeding as in Lam and Seberry (1984) and Seberry (1982, 1984) we now have:

**Theorem 6.1.** Whenever \(\lambda \equiv 0 \pmod{g}\), \(\lambda(u - 1) \equiv 0 \pmod{2}\), \(\lambda v(v - 1) \equiv 0 \pmod{6}\) for \(g\) odd or \(\lambda v(v - 1) \equiv 0 \pmod{24}\) for \(g\) even there exists a regular group divisible design with parameters \((v, g, r, 3, \lambda_1 = 0, \lambda_2 = \lambda/g, m = v, n = g)\).
References