ON BHASKAR RAO DESIGNS OF BLOCK SIZE FOUR

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Abstract. We show that Bhaskar Rao designs of type BRD(v, b, r, 4, 6) exist for v ≡ 0, 1 (mod 5) and of type BRD(v, b, r, 4, 12) exist for all v ≥ 4.

Let A, B and A + B be \( v \times b \) matrices with entries 0, 1. Then \( X = A - B \) is said to be a Bhaskar Rao design with parameters BRD(\( v, b, r, k, \lambda \)) when the following matrix equations are satisfied:

\[
XX^T = rI
\]

\[
(A + B)(A + B)^T = (r - \lambda)I + \lambda J
\]

\[
J(A + B) = kJ.
\]

\( X \) is a \( v \times b \) matrix with entries 0, +1, −1 with row inner product zero and which, when the −1 elements are replaced by +1, becomes the incidence matrix of a MBBD(\( v, b, r, k, \lambda \)). For example the following matrix is a BRD(6, 15, 10, 4, 6):

Example 1: There exists a BRD(6, 15, 10, 4, 6). Write—for −1

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

These designs were first studied by Bhaskar Rao [1, 2] and may be used to obtain group divisible PBIBD with parameters

\[
v^* = 2v, \quad b^* = 2b, \quad r^* = r, \quad k^* = k,
\]

\[
\lambda_1 = 0, \quad \lambda_2 = \lambda/2, \quad m = 2, \quad n = v.
\]
The necessary conditions for the existence of a BRD\((v, b, r, k, \lambda)\) are for \(k = 4\),
\[
\lambda(v-1) = r(k-1)
\]
\[
bd = vr
\]
and other restrictions on the parameters have been found (see [3, 4]) when \(k \neq 4\). They have also been studied by Vyas [5] and Singh [6]. We use the notation \(\text{BRD}(v, k, \lambda)\) for \(\text{BRD}(v, b, r, k, \lambda)\) as \(b\) and \(r\) are dependent on \(v, k, \lambda\).

In this paper we use the following known results (see [7]) restricted to the group \(Z_4\):

Theorem 1: Suppose there exists a \(\text{BRD}(k, j, \lambda B)\) and

(i) a \(\text{BRD}(v, k, \lambda A)\) then there exists a \(\text{BRD}(v, j, \lambda A \lambda B)\);

(ii) a \(\text{BIBD}(v, k, \lambda)\) then there exists a \(\text{BRD}(v, j, \lambda A \lambda B)\).

Or, as is obtained in a similar fashion:

Corollary 2: Suppose there exists a pairwise balanced design \(B(K, \lambda, v)\) where \(K = \{k_1, \ldots, k_i\}\) and a \(\text{BRD}(k_i, j, \mu)\) for each \(k_i \in K\) then there exists a \(\text{BRD}(v, j, \lambda \mu)\).

The next result is a slight improvement on the result of Lam and Seberry [7] where the existence of \(k - 1\) mutually orthogonal latin squares was required. The result may be proved by adjusting the matrix in the proof of the original theorem.

Theorem 3: Suppose there exists a \(\text{BRD}(v, k, \lambda)\) with a subdesign on \(w\) points (the values \(w = 0\) and \(1\) are allowed), a \(\text{BRD}(v, k, \lambda)\) and \(k - 2\) mutually orthogonal Latin squares then there exists a \(\text{BRD}(v(v-w)+w, k, \lambda)\) with subdesigns on \(v, w\) and \(v\) points.

Remark 4: In this paper we are interested in the case \(k = 4\), so we only need a pair of orthogonal latin squares and hence \(v - w\) may take on any value other than 2 or 6.

Hanani’s theorem stated on p. 260 of Hall [8] states

Theorem 5 (Hanani): Let \(u \equiv 0, 1 \pmod{5}\) then \(u \in B(K_4^1, 1)\) where \(K_4^1 = \{5, 6, 10, 11, 16, 20, 35, 36, 40, 70, 71, 75, 76\}\).

Remark: We now see that if \(u \equiv 0, 1 \pmod{5}\) and there exists a \(\text{BRD}(k_i, 4, 6)\) for every \(k_i \in K_4^1\) then we have the existence of a \(\text{BRD}(u, 4, 6)\) using either Theorem 1 or Corollary 2 with Theorem 5.

The main theorem: First we establish:

Theorem 6: Let \(p \geq 5\), odd, be a prime or prime power. Then there is a \(\text{BRD}(p, \frac{1}{2}p(p-1), 2(p-1), 4, 6)\).
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Proof: Let \( g \) be a generator of the multiplicative group, \( G \), of \( GF(p) \). Consider the initial sets, writing \( \bar{g}^a \) for \( g^a \) with the non-identity element of \( Z_2 \) attached,

\[
D_i = (\bar{g}, g', g'^{i+1}, g'^{i+2}) \text{ where } i = 0, 1, ..., \frac{1}{2}(p-3).
\]

The differences from \( D_i \) are

\[
E_i = (g', \bar{g}^{i+1}, \bar{g}^{i+2}, \bar{g}^{i(p-1)+i+1}, \bar{g}^{i(p-1)+i+2}, g^i(g-1), g^i(g-1), g^i(p-1)+i+1, g^i(p-1)+i+2, g^i(g-1), g^i(p-1)+i+1(g-1)).
\]

As \( i \) runs through \( 0, 1, ..., \frac{1}{2}(p-3) \) the totality of elements from all \( E_i \) including repetitions is 3 copies of \( G \) and 3 copies of \( \bar{G} \) with the non-identity element attached, that is,

\[
\sum_{i=0}^{\frac{1}{2}(p-3)} E_i = 3G + 3\bar{G},
\]

giving the result.

Theorem 7: A BRD(\( v, 4, 6 \)) exists for \( v \equiv 0, 1 \mod 5 \).

Proof: By the remark after Theorem 5 it is merely necessary to show the existence of a BRD(\( u, 4, 6 \)) for \( u \in K_4 \). These are obtained as given in Table 1 by developing the indicated initial blocks. First we exhibit a BRD(\( 8, 4, 6 \)):

Example 2: There is a BRD(\( 8, 28, 14, 4, 6 \)).
\[
B = \begin{bmatrix}
0 & 1 & - \\
- & 0 & 1 \\
1 & - & 0
\end{bmatrix}, \\
I = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

**Table 1.**

<table>
<thead>
<tr>
<th>treatment</th>
<th>construction</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Prime, Theorem 6.</td>
</tr>
<tr>
<td>6</td>
<td>Example 1.</td>
</tr>
<tr>
<td>7</td>
<td>Prime, Theorem 3.</td>
</tr>
<tr>
<td>8</td>
<td>Example 2.</td>
</tr>
<tr>
<td>9</td>
<td>Theorem 6.</td>
</tr>
<tr>
<td>10</td>
<td>((\infty, 0, 2, 4), (\infty, 0, 1, 3), (0, 1, 2, 5), (0, 5, 3, 6), (0, 1, 5, \bar{5}) \mod 9).</td>
</tr>
<tr>
<td>11</td>
<td>Prime, Theorem 6.</td>
</tr>
<tr>
<td>15</td>
<td>Form a BRD ((7, 4, 3)) by developing ((\bar{0}, 0, 1, 2, 4) \mod 7). Use this BRD with a BIBD ((15, 7, 3)) in Theorem 1 (ii).</td>
</tr>
<tr>
<td>16</td>
<td>((\infty, 0, 2, 6), (\infty, 0, 3, 8), (\bar{0}, 4, 5, 7), (\bar{0}, 4, 5, 7), (\bar{0}, 5, 7, 11), (\bar{0}, 1, 5, 7), (\bar{0}, 2, 3, 9), (\bar{0}, 1, 3, 6) \mod 15)).</td>
</tr>
<tr>
<td>20</td>
<td>((\infty, 0, 2, 6), (\infty, 0, 13, 16), (\bar{0}, 3, 5, 11), (\bar{0}, 3, 10, 16), (\bar{0}, 4, 5, 12), (\bar{0}, 2^{4+1}, 2^{5+2}, 2^{5+2}, 2^{5+2}) \mod 19).</td>
</tr>
<tr>
<td>55</td>
<td>35 = 5 \times 7, Theorem 3.</td>
</tr>
<tr>
<td>56</td>
<td>38 = 5(8 - 1) + 1, Theorem 3.</td>
</tr>
<tr>
<td>40</td>
<td>40 = 8 \times 5, Theorem 3.</td>
</tr>
<tr>
<td>70</td>
<td>70 = 7 \times 10, Theorem 3.</td>
</tr>
<tr>
<td>71</td>
<td>Prime, Theorem 6.</td>
</tr>
<tr>
<td>75</td>
<td>75 = 5 \times 15, Theorem 3.</td>
</tr>
<tr>
<td>76</td>
<td>76 = 19(6 - 1) + 1, Theorem 3.</td>
</tr>
</tbody>
</table>

Before we proceed to the case \(\lambda = 12\) we establish the existence of a few more BRD\((v, 4, 6)\):

**Theorem 8:** There exist BRD\((v, 4, 6)\) for \(v \in \{12, 14, 18, 22, 24, 32, 33, 38\}.


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**Proof:** Use Table 2 developing the initial blocks indicated.

**Table 2.**

<table>
<thead>
<tr>
<th>number of treatments</th>
<th>construction</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>((\infty, 0, 3, 6), (\infty, 0, 4, 5), (3, 4, 5, 9), (1, 3, 4, 5), (1, 3, 4, 5) \mod 11 \rangle )</td>
</tr>
<tr>
<td>14</td>
<td>((\infty, 0, 1, 3), (\infty, 0, 2, 5), (6, 2, 4, 8), (6, 1, 2, 6), (6, 1, 5, 6), (6, 1, 5, 6) \mod 13 \rangle )</td>
</tr>
<tr>
<td>18</td>
<td>((\infty, 0, 3, 4), (\infty, 0, 5, 7), (6, 1, 8, 13), (6, 5, 6, 8), (6, 5, 6, 8) \mod 17 \rangle )</td>
</tr>
<tr>
<td>22</td>
<td>((\infty, 0, 2, 6), (\infty, 0, 4, 10), (6, 5, 7, 16), (6, 4, 6, 14), (6, 1, 5, 10), (6, 1, 4, 9), (6, 1, 3, 16), (6, 3, 7, 18) \mod 21 \rangle )</td>
</tr>
<tr>
<td>24</td>
<td>((\infty, 0, 4, 12), (\infty, 0, 4, 10), (6, 1, 2, 6), (6, 1, 2, 6), (6, 1, 2, 6) \mod 23 \rangle )</td>
</tr>
<tr>
<td>32</td>
<td>((\infty, 0, 2, 6), (\infty, 0, 1, 8), (6, 3, 4, 15), (6, 1, 6, 14), (6, 3, 7, 9), (6, 9, 12, 27), (6, 3, 4, 5, 7, 8, 10, \ldots, 14) \mod 31 \rangle )</td>
</tr>
<tr>
<td>33</td>
<td>(33 = 8(5-1)+1, \text{Theorem } 3.)</td>
</tr>
<tr>
<td>38</td>
<td>((\infty, 0, 2, 12), (\infty, 0, 1, 17), (6, 4, 17, 33), (6, 10, 12, 13), (6, 3, 6, 16), (6, 4, 11, 12), (6, 2, 2, 12, 2, 12, 2), (6, 1, 4, 5, 7, 8, 9, 11, \ldots, 17) \mod 37 \rangle )</td>
</tr>
</tbody>
</table>

We use Hanani's theorem quoted from Hall [8, p. 248] to establish the result for \(\text{BRD}(v, 4, 12)\).

**Theorem 9 (Hanani):** Let \(u \geq 4\) then \(u \in B(K^3_v, 1)\) where

\[
K^3_v = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 22, 23, 27\}.
\]

Now we can show

**Theorem 10:** An \(\text{BRD}(v, 4, 12)\) exists for all \(v \geq 4\).

**Proof:** We note any four distinct rows of an Hadamard matrix of order 12 gives the result for \(v = 4\). Thus, just as in the remark after Theorem 5, it is merely necessary to show the existence of a \(\text{BRD}(u, 4, 12)\) for \(u \in K^3_v\).

Taking two copies of the \(\text{BRD}(u, 4, 6)\) given above gives the result immediately.
REFERENCES


