Maximal Ternary Codes and Plotkin's Bound

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The analogue of Plotkin's bound is developed for ternary codes with high distance relative to length, obtaining

$$A(n,d) \leq 3 \left[ \frac{d}{3d-2n} \right] \quad 3d > 2n,$$

$$A(n,2n/3) = 3n \quad 3d = 2n.$$

Generalized Hadamard matrices are used to obtain codes which meet these bounds. The ternary analogue of Levenshtein's construction is discussed and maximal codes constructed.

The Plotkin bound.

Let $A$ be the $M \times n$ matrix whose rows are the codewords of a ternary code, $C$. Suppose the $i$th column of $A$ contains $x_i 0's$, $y_i 1's$, $z_i 2's$. Following the method of Plotkin we calculate the sum

$$\sum_{u \in C} \sum_{v \in C} \text{dist}(u,v)$$

in two ways. Since $\text{dist}(u,v) \geq d$ if $u \neq v$, the sum is $\geq M(M-1)d$.

But each column of $A$ contributes

$$2x_iy_i + 2y_iz_i + 2z_ix_i, \quad x_i + y_i + z_i = M$$

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to the above sum. Hence summing over all columns we have

\[ S = \sum_{i=1}^{0} (2x_1y_1 + 2y_1z_1 + 2z_1x_1) \geq M(M-1)d. \]

We use Lagrange multipliers to maximize

\[ S + \sum_{i=1}^{0} (x_iy_i + z_i - M) \]

and find the maximum occurs when \( x_1 = y_1 = z_1 = M/3 \) and so if

a) \( M \equiv 0 \pmod{3} \) we have

\[ M(M-1)d \leq 2M^2n/3 \]

\[ M \leq \frac{3d}{3d-2n} \]

and since \( 3|M \) we have

\[ M \leq 3 \left[ \frac{d}{3d-2n} \right] \quad \text{for} \quad 3d > 2n. \quad (1) \]

b) \( M = t+3s, t \neq 3 \), the maximum occurs when \( t \) of \( x_1, y_1, z_1 \) are

\( s+1 \) and \( 3-t \) are \( s \) and so

\[ M(M-1)d \leq n(4s(s+1) + 2(s+t-1)^2). \]

Now \( t = 1 \) gives

\[ 3s(3s+1)d \leq 2ns(3s+2) \]

or

\[ s \leq \frac{4n-3d}{9d-6n} \quad \text{as} \quad s \geq 0 \]

and as \( M \equiv 1 \pmod{3} \)

\[ M \leq 3 \left[ \frac{4n-3d}{9d-6n} \right] + 1 \quad \text{for} \quad 3d > 2n. \]

Also \( t = 2 \) gives

\[ (3s+2)(3s+1)d \leq 2n(3s+1)(s+1) \]

or

\[ s \leq \frac{2n-2d}{3d-2n} \quad \text{as} \quad 3s+1 \geq 0 \]

and as \( M \equiv 2 \pmod{3} \)

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\[ M \leq 3 \left\lfloor \frac{2n-2d}{3d-2n} \right\rfloor + 2 \]
\[ \leq 3 \left\lfloor \frac{d}{3d-2n} \right\rfloor - 1 \quad \text{for } 3d > 2n. \]

c) If \( 2n = 3d \) we have \( M = 3n \).

Summarizing, we have, noting (1) is larger than the other bounds:

**Theorem 1.** For an alphabet of 3 symbols, the maximum number of codewords of length \( n \) and distance \( d \), \( A(n,d) \), satisfies

\[ A(n,d) \leq 3 \left\lfloor \frac{d}{3d-2n} \right\rfloor \quad \text{for } 3d > 2n \geq 2d. \]\( (2) \)

This result is given by Blake and Mullin (p 65) but also

**Lemma 1.** \( A(n,d) \leq 3A(n-1,d) \).

**Proof.** Given a ternary \((n,M,d)\) code, the codewords fall into three classes, those beginning with 0, 1 and 2. One class must contain at least one third of the codewords, thus

\[ A(n-1,d) \geq A(n,d)/3. \]

This gives us

**Theorem 2.** (i) \( A(3n,2n) \leq 9n. \)

(ii) \( A(3n+1,2n+1) \leq 6n+3. \)

(iii) \( A(3n+1,2n) \leq 27n. \)

(iv) \( A(3n-1,2n) \leq 3n. \)

**Proof.** (i) \( A(3n,2n) \leq 3A(3n-1,2n) \leq 3 \left\lfloor \frac{2n}{6n-2(3n-1)} \right\rfloor = 9n. \)

(ii) \( A(3n+1,2n+1) \leq 3 \left\lfloor \frac{2n+1}{3(2n+1)-2(3n+1)} \right\rfloor = 6n+3. \)
(iii) \( A(3n+1, 2n) \leq 3A(3n, 2n) = 27n. \)

(iv) \( A(3n-1, 2n) \leq 3\left[\frac{2n}{6n - 2(3n-1)}\right] = 3n. \)

To construct maximal ternary codes we require some lemmas concerning generalized Hadamard matrices of the form \( G(n, Z_3) \). This paper will not discuss the theory of generalized Hadamard matrices, nor their existence or non-existence. However, the following definitions and lemmas are required.

A square matrix of size \( n \) with entries from a group \( G \) is called a generalized Hadamard matrix, \( G(n, G) \), if the inner product of any two distinct rows, \( a = (a_1, \ldots, a_n) \), \( b = (b_1, \ldots, b_n) \), \( a_i b_j \in G \), defined by \( a \cdot b = \sum_{i=1}^{n} a_i b_i^{-1} \) is \( n/|G| \) copies of \( G \). For example we have

\[
G(3, Z_3) = \begin{pmatrix}
1 & 1 & 1 \\
1 & w & w^2 \\
1 & w^2 & w
\end{pmatrix}
\]

\[
G(6, Z_3) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & w & w^2 & w^2 & w \\
1 & w & w & w^2 & w^2 \\
1 & w^2 & w & 1 & w \\
1 & w^2 & w^2 & w & 1 \\
1 & w & w^2 & w^2 & w
\end{pmatrix}
\]

The results of de Launey, Drake, Junghckel, Rajkundi, Seberry, Seiden and Street allow us to say:

**Lemma 2.** There exist \( G(n, Z_3) \) for

(i) \( n = 3^k \), (ii) \( n = 6 \), (iii) \( n = 12 \), (iv) \( 2, 3, 4, 5 \). A \( G(n, Z_3) \) does not exist for \( n = 15, n = 21 \) is undecided.

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Any $GH(n, 3)$ is equivalent to a $GH(n, 2)$ with its first row and column consisting entirely of the unit element of the group.

**Lemma 3.** A $GH(n, 3)$ gives block codes over a 3-symbol alphabet with parameters, $(n, k, d)$:

(i) $(n, 3n, 2n/3)$, (ii) $(n-1, 3n, 2n/3-1)$, (iii) $(n-1, n, 2n/3)$,
(iv) $(n-2, n, 2n/3-1)$.

**Lemma 4.** The following codes exist over a 3-symbol alphabet:

(i) $(4, 9, 3)$, (ii) $(5, 18, 3)$, (iii) $(8, 17, 5)$.

<table>
<thead>
<tr>
<th>(4,9,3)</th>
<th>(5,18,3)</th>
<th>(8,17,5)</th>
</tr>
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<tbody>
<tr>
<td>0 0 0 0</td>
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<tr>
<td>0 1 2 1</td>
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<td>2 2 2 2</td>
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<td>0 2 1 2</td>
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<td>0 1 1 0</td>
<td>1 1 0 1</td>
<td></td>
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</tbody>
</table>

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A number of authors including de Launey, Lam, Seberry and Street and Rodger have studied an extension of generalized Hadamard matrices in which the elements are over a group ring, called Bhaskar Rao designs (BRD). Since we are concerned with ternary codes we restrict ourselves to the group ring \( \{0\} + Z_2 \). A Bhaskar Rao design \( W = A - B \) with parameters \( v,b,r,k,\lambda \) satisfies
\[
WT = rI_v, \\
(A+B)(A+B)^T = (r-\lambda)I + \lambda J \\
J(A+B) = kJ, (A+B)J = rJ,
\]
where \( A, B \) are \( (0,1) \)-matrices, \( A+B \) is a BRD\((v,b,r,k,\lambda)\). The BRD is written \( \text{BRD}(v,b,r,k,\lambda;Z_2) \) or \( \text{BRD}(v,k,\lambda) \) for brevity. Such designs can be extended to partially balanced and pairwise balanced designs and to groups other than \( Z_2 \).

In the remainder of this section we use
\[
r = \lambda(v-1)/(k-1) \\
b = vr/k,
\]
and if \( W = A - B \) is a BRD where \( A^J = JA = k_1 J, B^J = JB = k_2 J \)
\[
k = k_1 + k_2 \\
\lambda(v-1)/2 = k_1(k_1-1) + k_2(k_2-1).
\]

**Lemma 5.** If there exists a BRD\((v,b,r,k,\lambda)\) then there exist 3-symbol codes with parameters
\[
(i) \ (vr/k,v,2r-3\lambda/2), \\
(ii) \ (vr/k,v+3,\min(2r-3\lambda/2,b-k_1,b-k_2,b-r)), \\
(iii) \ (vr/k,2v,\min(r,2r-3\lambda/2)).
\]

**Proof.** Let \( X \) be the BRD. The result follows by considering the rows of
Corollary 6. Since there exist BRD(7,4,2), BRD(13,9,6), BRD(19,9,6), BRD(21,16,12), BRD(8,28,14,4,6) and BRD(13,26,8,4,2) there exist 
(7,7,5), (7,10,5), (7,14,4), (13,25,9), (19,19,12), (19,38,9), 
(21,21,14), (21,42,14), (28,8,19), (28,16,14), (26,13,13), (26,26,8) 
codes.

Lemma 7. If there exists a regular (= constant row sum) BRD(v,k,\lambda) 
then there exist 3-symbol codes with parameters (v\chi/k,\lambda,\min(2\chi-3\lambda/2, 
x_1+2\chi-2\lambda),-x_1+b-\lambda/2) where \chi is the number of occurrences of 
[0 1] 
and 0 \leq x_1 \leq v-\lambda. The regular cyclic BRD(t^2+t+1,t^2,t^2-t) 
gives 
\{t^2+t+1,3(t^2+t+1),5(t^2+t+2)\}-codes.

Proof. Let M be the BRD and consider 
\[
\begin{pmatrix}
N \\
M+1 \\
M+2
\end{pmatrix}
\]

Any two rows of this code, \(a, b\), can be written as 
\[
a = 0 \quad \ldots \quad 01 \quad \ldots \quad 12 \quad \ldots \quad 2 \\
b = 1 \ldots 12 \ldots 20 \ldots 01 \ldots 12 \ldots 20 \ldots 0 \\
x_1 \quad y_1 \quad z_1 \quad x_2 \quad y_2 \quad z_2 \quad x_3 \quad y_3 \quad z_3
\]

Without loss of generality we assume \(k_1 \geq k_2\). Now 
\[
k_1 = x_2 y_2 z_2 = x_1 x_2 x_3 (1), \quad k_2 = y_3 y_2 z_3 = y_1 y_2 y_3 (2), \quad r = x_1 (3)
\]
assuming regularity of rows. Now the fact that the rows of the BRD (when 
considered as 0, 1) are orthogonal means
\[ x_2 + y_3 = y_2 + x_3 = \frac{\lambda}{2} \quad (4) \]

and we have
\[ x_1 + y_1 = r - \lambda = z_2 + z_3, \quad (5) \]
\[ z_1 = b - 2r + \lambda. \quad (6) \]

Considering \(a, b, c = \frac{1}{4}\) and \(b + \frac{1}{2}\) as codewords we want the minimum of the distances \(d_1 = d(a, b), d_2 = d(a, b + \frac{1}{2}), d_3 = d(a, b + 1)\) over all \(a, b\) in the code. Now
\[ d_1 = x_1^4 + x_2^4 + x_3^4 + 2x_1 \cdot x_2 \cdot x_3^2 = 2(x_1^2 + x_2^2 + x_3^2) = 2x - 3\lambda/2 \]
\[ d_2 = x_1^2 + x_2^2 + x_2 \cdot x_3 + 2x_3^2 = x_1^2 + x_2^2 + x_3^2 = 2b + 2r + 2\lambda \]
\[ d_3 = y_1^2 + x_2^2 + x_2 \cdot x_3 + 2x_3^2 = 2b - d_1 = d_2 = -x_1 + b - \lambda/2. \]

Hence distance of code is \(\min(2r - 3\lambda/2, 2x_1 + b - 2r + 2\lambda, -x_1 + b - \lambda/2)\) where \(0 \leq x_1 \leq \min(k_1, r - \lambda)\) is the number of occurrences of \(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\).

In particular for cyclic BBD(t^2 + t + 1, t^2 - t) the minimum distance is \(4(t^2 + t + 2)\).

\[ \square \]

**Corollary 8.** There exist \((7, 21, 4), (13, 39, 7), (19, 57, 12)\) and \((21, 63, 11)\) codes.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(M(\text{found}))</th>
<th>(M(\text{max}))</th>
<th>(d)</th>
<th>Construction or Comment</th>
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<td>9</td>
<td>2</td>
<td>Lemma 3 (i)</td>
</tr>
<tr>
<td>4</td>
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<td>3</td>
<td>Lemma 4 (i)</td>
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<td>Lemma 3 (i)</td>
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<td>7</td>
<td>10</td>
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<td>5</td>
<td>Corollary 6</td>
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<td>Lemma 1 and 2 (iii)</td>
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<td>27</td>
<td>27</td>
<td>6</td>
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<tr>
<td>9</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>(4, 9, 3) + (5, 6, 4)</td>
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<tr>
<td>10</td>
<td>12</td>
<td>18</td>
<td>7</td>
<td>From (11, 12, 9), Lemma 1</td>
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<tr>
<td>10</td>
<td>6</td>
<td>6</td>
<td>8</td>
<td>(5, 6, 4) + (5, 6, 4)</td>
</tr>
</tbody>
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Lemma 9. Let \( a \) and \( b \) be two ternary vectors. Then

\[
\begin{align*}
d(a, b) + d(a, b + 1) + d(a, b + 2) &= 2n \\
d(a + i, b + j) &= d(a + i\xi, b + j\xi), i, j, \xi \in \{0, 1, 2\}.
\end{align*}
\]

Proof. The second part of the lemma is obviously true for linear codes but we show it is also true for block codes. We write the two codewords as

\[
\begin{align*}
a &= 0 & \ldots & 01 & \ldots & 12 & \ldots & 2 \\
b &= 0 & \ldots & 01 & \ldots & 12 & \ldots & 2 \\
&= x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} & x_{20} & x_{21} & x_{22}
\end{align*}
\]

Now

\[
\begin{align*}
d(a, b) &= x_0 + x_2 + x_4 + x_6 + x_8 + x_{10} + x_{12} + x_{14} + x_{16} + x_{18} + x_{20} + x_{22} = d(a + 1, b + 1) - d(a + 2, b + 2) \\
d(a + 1, b) &= x_0 + x_2 + x_4 + x_6 + x_8 + x_{10} + x_{12} + x_{14} + x_{16} + x_{18} + x_{20} + x_{22} = d(a, b + 2) - d(a + 2, b + 1) \\
d(a + 2, b) &= x_0 + x_2 + x_4 + x_6 + x_8 + x_{10} + x_{12} + x_{14} + x_{16} + x_{18} + x_{20} + x_{22} = d(a, b + 1) - d(a + 1, b + 2) \\
\end{align*}
\]

Further

\[
\begin{align*}
d(a, b) + d(a, b + 1) + d(a, b + 2) &= 2 \sum_{i=0}^{1} \sum_{j=0}^{3} x_{ij} = 2n. 
\end{align*}
\]
This allows us to readily test the distance of a constructed code, as in the following.

**Lemma 10.** Suppose \( A \) is a ternary \((n,M,d)\)-code then
\[
\begin{bmatrix}
A \\
A + 1 \\
A + 2
\end{bmatrix}
\]
is a ternary \((n,3M,d')\)-code where \( d' = \min \{ d(a,b), d(a,b+1), 2n-d(a,b) \} \). \( a, b \) are codewords of \( A \).

**Example 1.** Suppose \( A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{bmatrix} \), a ternary \((5,6,4)\)-code, then \( a = (0,0,0,0,0) \) with \( b = (0,1,2,2,1) \) which gives \( d' = \min(4,3,3) = 3 \) and \( a = (0,1,2,2,1) \) with \( b = (2,1,0,1,2) \) which gives \( d' = \min(4,3,3) = 3 \) giving a \((5,10,3)\)-code.

**Theorem 3.** Let \( A \) be a ternary \((n,1,M,d)\)-code. Further define
\[
d_1 = \min_{a \in \mathbb{A}} d(a,b+1) \quad \text{and} \quad d_2 = 2(n-1)-d-d_-.
\]
Then there exists ternary \((n,3M,\min(d_1,d_2+1))\) and \((n-1,3M,\min(d_1,d_2))\) codes.

**Proof.** We consider
\[
\begin{bmatrix}
0 \\
\vdots \\
1 \\
A \\
A + 1 \\
\vdots \\
A + 2 \\
2
\end{bmatrix} = \begin{bmatrix}
N_0 \\
N_1 \\
N_2
\end{bmatrix}
\]

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Any two rows of \( N_0 \) have distance \( \geq d \). By Lemma 9 any two rows of \( N_1 \) and \( N_2 \) also have distance \( \geq d \). By the second part of Lemma 9 the distance between any two rows of \( A_i \) and \( A_j, i \neq j \), is \( d_1 \) or \( d_2 \) where \( d_1 d_2 = 2(n-1) \). Hence the minimum distance between rows of \( N_1 \) and \( N_j, i \neq j \), is \( d_1+1 \) or \( d_2+1 = 2(n-1)-d-d_1+1 \). This gives the result.

Example 2.

\[
A = \begin{bmatrix}
2 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 2 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 2 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 2 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 2 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 2 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 2
\end{bmatrix}
\]

is a \((7,7,5)\)-code with constant distance \( d = 5 \). Now \( d_1 = 3 \) and \( d_2 = 6 \). So in the Theorem

\[
6n = 46 = 3(5+4+7)
\]

giving a \((7,21,3)\) and \((8,21,4)\)-code.

Example 3. For the \((13,13,9)\)-code with constant distance \( d = 9 \),
\( d_1 = 10 \) and \( d_2 = 7 \). So there exist \((13,39,7)\) and \((14,39,8)\)-codes.

We can sometimes do better. We use

\[
d_0 = \min d(a,b), \quad d_1 = \min d(a,b+1), \quad d_2 = \min d(a,b+2),
\]

where \( a,b \) run over all vectors of the code.

THEOREM 6. Let \( A \) be a ternary \((n,M,d)\) code. Then there are ternary codes with parameters:

i) \((2n,2M,\min(2d,2n-2)), (2n,3M,\min(2d,2n-d,2d_1))\)

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ii) \((3n, 9M, \min(n, 2d, d+d_1, 2n-d_1, 2n-d))\),

iii) \((3n, 9M, \min(3d, 3d_1, 2n-d, 2n-d_1))\).

**Proof.**

i) With \(d_0, d_1, d_2\) defined above, consider

\[
\begin{bmatrix}
A & A \\
A+1 & A+2
\end{bmatrix}
\begin{bmatrix}
A & A \\
A+1 & A+2
\end{bmatrix}
\begin{bmatrix}
A & A \\
A+1 & A+2
\end{bmatrix}
\begin{bmatrix}
A & A \\
A+1 & A+2
\end{bmatrix}
\begin{bmatrix}
A & A \\
A+1 & A+2
\end{bmatrix}
\begin{bmatrix}
A & A \\
A+1 & A+2
\end{bmatrix}
\]

The distances are \(2d, d_1+d_2 = 2n-d\) respectively.

ii) Consider

\[
\begin{bmatrix}
A & A \\
A & A+1 \\
A & A+2 \\
A+1 & A \\
A+1 & A+1 \\
A+1 & A+2 \\
A+2 & A \\
A+2 & A+1 \\
A+2 & A+2
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 1 \\
0 & 2 \\
1 & 0 \\
1 & 1 \\
1 & 2 \\
2 & 0 \\
2 & 1 \\
2 & 2
\end{bmatrix}
\]

The distances are \(2d, n, n+d_1, n+d_2, n+d, d+d_1, d+d_2, d_1+d_2, 2n\).

Since \(d_1 < n\), we only need to consider \(n, 2d, d_1, 2n-d_1, 2n-d\).

iii) Let \(B\) be the \((3,9,2)\) ternary code. Consider \(B \ominus A\)

\[
\begin{bmatrix}
\mathbf{0} & \mathbf{0} \\
\mathbf{1} & \mathbf{2} \\
\mathbf{2} & \mathbf{1}
\end{bmatrix}
\]

where \(\mathbf{0}\) is defined in (ii). The distances are \(3d, d_1+d_2, d_1, d_1+d_3, 3n, 3d_1, i, j = 1, 2, 3\).

Now \(d_1 < n\) and \(d_1+d_3 = 2n\), so we only need \(3d, 2n-d, 2n-d_1, 3d_1, 6n-3d-3d_1\).

**Example 4.** Use \(A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}\) which is a \((2,3,2)\)-code. Then \(B\) in

\[
\begin{bmatrix}
0 & 0 \\
1 & 2 \\
2 & 1
\end{bmatrix}
\]

is a \((4,37,2)\)-code since \(d = 2, n = 3\). \(C\) is a \((6,27,3)\)-code.
\[
C = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
1 & 2 & 1 & 2 & 1 \\
2 & 1 & 2 & 1 & 2 \\
0 & 0 & 1 & 1 & 2 \\
1 & 2 & 2 & 0 & 1 \\
2 & 1 & 0 & 2 & 1 \\
0 & 0 & 2 & 2 & 1 \\
1 & 2 & 6 & 1 & 2 \\
2 & 1 & 1 & 6 & 0 \\
1 & 1 & 0 & 0 & 2 \\
1 & 0 & 1 & 2 & 0 \\
0 & 2 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
2 & 0 & 2 & 0 & 2 \\
0 & 2 & 0 & 2 & 0 \\
1 & 1 & 2 & 2 & 0 \\
2 & 0 & 0 & 1 & 2 \\
0 & 2 & 1 & 0 & 2 \\
2 & 2 & 0 & 0 & 1 \\
0 & 1 & 1 & 2 & 2 \\
1 & 0 & 2 & 1 & 0 \\
2 & 2 & 1 & 1 & 0 \\
0 & 1 & 2 & 0 & 1 \\
1 & 0 & 0 & 2 & 1 \\
2 & 2 & 2 & 2 & 2 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 
\end{bmatrix}
\]

**Example 5.** Using the \((5,6,4)\)-code of Example 1 we have \((10,18,6)\) and \((10,54,5)\) ternary codes.

**Corollary 11.** There exist ternary \((2^{t-1},3^t,2^t)\)-codes \(t \geq 1\).

**Proof.** Use the Theorem with the \((4,27,2)\)-code of Example 4.
<table>
<thead>
<tr>
<th>$n$</th>
<th>$M(\text{found})$</th>
<th>$M(\text{max})$</th>
<th>$d$</th>
<th>construction</th>
</tr>
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<tbody>
<tr>
<td>3</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>Lemma 3</td>
</tr>
<tr>
<td>4</td>
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<td>2</td>
<td>2</td>
<td>Example 4</td>
</tr>
<tr>
<td>5</td>
<td>18</td>
<td>3</td>
<td>3</td>
<td>Lemma 4</td>
</tr>
<tr>
<td>6</td>
<td>27</td>
<td>3</td>
<td>3</td>
<td>Example 4</td>
</tr>
<tr>
<td>6</td>
<td>18</td>
<td>18</td>
<td>4</td>
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</tr>
<tr>
<td>7</td>
<td>21</td>
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<td>4</td>
<td>Corollary 6</td>
</tr>
<tr>
<td>8</td>
<td>27</td>
<td>5</td>
<td>5</td>
<td>From $(9,27,6)$-code</td>
</tr>
<tr>
<td>9</td>
<td>27</td>
<td>6</td>
<td>6</td>
<td>Lemma 3</td>
</tr>
<tr>
<td>10</td>
<td>54</td>
<td>5</td>
<td>5</td>
<td>Example 5</td>
</tr>
<tr>
<td>10</td>
<td>18</td>
<td>6</td>
<td>6</td>
<td>Twice $(5,18,3)$-code</td>
</tr>
<tr>
<td>11</td>
<td>36</td>
<td>7</td>
<td>7</td>
<td>From $(12,36,8)$-code</td>
</tr>
<tr>
<td>12</td>
<td>36</td>
<td>8</td>
<td>8</td>
<td>Lemma 3</td>
</tr>
<tr>
<td>13</td>
<td>39</td>
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<td>9</td>
<td>Corollary 6</td>
</tr>
<tr>
<td>14</td>
<td>39</td>
<td>8</td>
<td>8</td>
<td>Example 3</td>
</tr>
<tr>
<td>15</td>
<td>18</td>
<td>10</td>
<td>10</td>
<td>$(6,18,4) + (9,27,5)$</td>
</tr>
<tr>
<td>16</td>
<td>54</td>
<td>10</td>
<td>10</td>
<td>From $(17,54,10)$-code</td>
</tr>
<tr>
<td>17</td>
<td>54</td>
<td>11</td>
<td>11</td>
<td>From $(18,54,12)$-code</td>
</tr>
<tr>
<td>18</td>
<td>54</td>
<td>12</td>
<td>12</td>
<td>Lemma 3</td>
</tr>
<tr>
<td>19</td>
<td>57</td>
<td>12</td>
<td>12</td>
<td>Corollary 6</td>
</tr>
<tr>
<td>20</td>
<td>21</td>
<td>13</td>
<td>13</td>
<td>From $(21,21,14)$-code</td>
</tr>
<tr>
<td>21</td>
<td>21</td>
<td>14</td>
<td>14</td>
<td>Corollary 6</td>
</tr>
<tr>
<td>22</td>
<td>36</td>
<td>15</td>
<td>14</td>
<td>Twice $(11,36,7)$-code</td>
</tr>
<tr>
<td>23</td>
<td>36</td>
<td>15</td>
<td>15</td>
<td>From $(24,36,16)$-code</td>
</tr>
<tr>
<td>24</td>
<td>36</td>
<td>16</td>
<td>16</td>
<td>Twice $(12,36,8)$-code</td>
</tr>
<tr>
<td>25</td>
<td>81</td>
<td>16</td>
<td>16</td>
<td>From $(26,81,17)$-code</td>
</tr>
<tr>
<td>26</td>
<td>81</td>
<td>17</td>
<td>17</td>
<td>From $(27,81,18)$-code</td>
</tr>
<tr>
<td>27</td>
<td>81</td>
<td>18</td>
<td>18</td>
<td>Lemma 3</td>
</tr>
<tr>
<td>28</td>
<td>36</td>
<td>19</td>
<td>19</td>
<td>From $(29,36,20)$-code</td>
</tr>
<tr>
<td>29</td>
<td>36</td>
<td>20</td>
<td>20</td>
<td>From $(30,36,20)$</td>
</tr>
<tr>
<td>30</td>
<td>36</td>
<td>20</td>
<td>20</td>
<td>From $(12,36,8) + (18,54,12)$</td>
</tr>
</tbody>
</table>

Table 2

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Levenshtein's Method:

Let us suppose that an arbitrary \( GH(m, Z_4) \) exists, written on the additive group, whose first column is composed entirely of zero's: denote this matrix by \( M_0 \); and the matrix when is formed by stripping off the column of zero's by \( M_0' \).

**Example 6.** \( GH(6, Z_4) = M_6 \)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 2 & 1 \\
0 & 1 & 0 & 1 & 2 & 2 \\
0 & 2 & 1 & 0 & 1 & 2 \\
0 & 2 & 2 & 1 & 0 & 1 \\
0 & 1 & 2 & 2 & 1 & 0 \\
\end{bmatrix}
\]

\[ M_6' = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 2 & 1 & 0 \\
1 & 0 & 1 & 2 & 2 & 1 \\
2 & 1 & 0 & 1 & 2 & 2 \\
2 & 2 & 1 & 0 & 1 & 2 \\
1 & 2 & 2 & 1 & 0 & 0 \\
\end{bmatrix} \]

The theory giving the construction of maximal codes requires matrices of particular orders and distances. The proofs of the following two lemmas are obvious:

**Lemma 12.** If there exists an \( M_{3t} \) (respectively \( M_{3(t+1)} \)) then the rows of \( M_{3t}' \) (respectively \( M_{3(t+1)}' \)) form a code with parameters \( n = 3t-1, M = 3t, d = 2t \) (respectively \( n = 3t+2, M = 3(t+1), d = 2(t+1) \)).

Write

\[
i = \begin{bmatrix}
\frac{d}{3d-2n} \\
\frac{d}{3d-2n}
\end{bmatrix}
\]

**Lemma 13.** If \( 3d > 2n \geq 2d \) then there exist integers \( a \) and \( b \) such that

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Proof. We can define \( i \) in terms of the following inequalities

\[
\left[ \frac{d}{3d-2n} \right] - 1 < i \leq \left[ \frac{d}{3d-2n} \right]
\]

that is

\[
\frac{2n-2d}{3d-2n} < i \leq \frac{d}{3d-2n}.
\]

Consider the left inequality

\[
2n-2d < i(3d-2n)
\]

which on rearranging gives

\[
\frac{2(i+1)}{(3i+2)} < \frac{d}{n}.
\]

Consider the right inequality of (5)

\[
i(3d-2n) \leq d
\]

which on rearranging gives

\[
\frac{d}{n} \leq \frac{2i}{(3i+1)}.
\]

Combining (6) and (7) we obtain

\[
\frac{2(i+1)}{(3i+2)} < \frac{d}{n} \leq \frac{2i}{(3i+1)}.
\]

The two inequalities of (8) may be written in determinant form:

\[
\begin{vmatrix}
  d & 2(i+1) \\
  n & (3i+2)
\end{vmatrix} > 0 = A \text{ say}
\]

\[
\begin{vmatrix}
  n & (3i-1) \\
  d & 2i
\end{vmatrix} \geq 0 = B \text{ say}.
\]
Now suppose that both A and B are even. Then let

\[ A = 2a \]
\[ B = 2b \]

so (9) and (10) become

\[ A = 2a = 2d(3i+2) - 2n(i+1) \]  \hspace{1cm} (11)
\[ B = 2b = 2ni - d(3i-1) \]  \hspace{1cm} (12)

solving (11) and (12) for a and d yield the required results (3).

Note: requiring that A and B are even imposes only one condition, namely that d is also even, but in the case of ternary codes the distance is in fact even for maximal codes as then 3d = 2n.

Following Levenshtein, we define the operation of adjunction of the matrices

\[ X = \begin{pmatrix}
X_{11} & X_{12} & \cdots & X_{1n_1} \\
X_{21} & X_{22} & \cdots & X_{2n_1} \\
\vdots & \vdots & \ddots & \vdots \\
X_{L1} & X_{L2} & \cdots & X_{Ln_1}
\end{pmatrix} \]

and

\[ Y = \begin{pmatrix}
Y_{11} & Y_{12} & \cdots & Y_{1n_2} \\
Y_{21} & Y_{22} & \cdots & Y_{2n_2} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{L1} & Y_{L2} & \cdots & Y_{Ln_2}
\end{pmatrix} \]

as follows

\[ X + Y = \begin{pmatrix}
X_{11} & X_{12} & \cdots & X_{1n_1} & Y_{11} & Y_{12} & \cdots & Y_{1n_2} \\
X_{21} & X_{22} & \cdots & X_{2n_1} & Y_{21} & Y_{22} & \cdots & Y_{2n_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
X_{L1} & X_{L2} & \cdots & X_{Ln_1} & Y_{L1} & Y_{L2} & \cdots & Y_{Ln_2}
\end{pmatrix} \]
where \( L = \min(L_1, L_2) \).

The operation of extension of a matrix \( X \), \( r \) times, is defined as the result of the consecutive adjunction of \( r \) matrices \( X \).

Note: If the rows of the matrix \( X \) form a code with the parameters \( n_1, d_1 \) and \( M_1 \), and the rows of a matrix \( Y \) form a code with the parameters \( n_2, d_2 \) and \( M_2 \), then the rows of the matrix \( aX + bY \) where \( a \) and \( b \) are integer non-negative numbers, form a code with the parameters

\[
\begin{align*}
    n &= an_1 + bn_1 \\
    d &= ad_1 + bd_2 \\
    M &= \min(M_1, M_2)
\end{align*}
\]

**Theorem 5.** If \( d \) is even and \( 3d > 2n > 2a \) then the following matrix \( M \) is maximal in that it meets the bound

\[
A(n,d) = 3 \left[ \frac{d}{3d-2n} \right] + 3i
\]

\[M = aM_{3i} + bM_{3(i+1)} \]

\[a = \frac{4d(3i+2) - n(i+1)}{n - \frac{4d(3i-1)}{3}}\]

**Proof.** For the proof of the theorem, it is sufficient to see, using lemmas 12 and 13 that the above construction does indeed generate a maximal code.

**Example 7.** As an illustration, we present the maximal \((13,6,10)\) code
Table 3  Examples of maximal ternary codes constructed using Theorem 3.

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>d</th>
<th>i = ⌊d/(3d-2n)⌋</th>
<th>a</th>
<th>b</th>
<th>Construction</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>M₆'</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>6</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>M₃' + M₆'</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>6</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>M₉'</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>8</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2M₃' + M₆'</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>8</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2M₆'</td>
</tr>
<tr>
<td>11</td>
<td>12</td>
<td>8</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>M₁₂'</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>10</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>3M₃' + M₆'</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>M₃' + 2M₆'</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>10</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>M₆' + M₉'</td>
</tr>
<tr>
<td>13</td>
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<td>12</td>
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<td>4</td>
<td>1</td>
<td>4M₃' + M₆'</td>
</tr>
<tr>
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<td>5</td>
<td>1</td>
<td>0</td>
<td>M₁₅'</td>
</tr>
</tbody>
</table>

Note: the code does not exist since M₁₅ does not exist.

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References


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