Constructions are given for generalised Hadamard matrices and weighing matrices with entries from abelian groups.

These are then used to construct families of SBIBD giving alternate proofs to those of Rajkundalia.

1. DEFINITION

A generalised Hadamard matrix \(GH(n,G)\) is an \(n \times n\) matrix with elements from the abelian group \(G\) of order \(|G|\) such that if \(a = (a_1, \ldots, a_n)\) and \(b = (b_1, \ldots, b_n)\) are any two rows of \(GH(n,G)\) then the elements \(a_i b_j^\alpha, i=1, \ldots, n\) give \(n/|G|\) copies of \(G\).

These matrices were considered by Butson [4,5], by Shrikhande [18] in connection with combinatorial designs, by Delarte and Goethals [6,7] in connection with codes and Bose [8] in connection with \(\lambda\)-geometries.

A generalised weighing matrix \(GW(n,k,G)\) is an \(n \times n\) matrix with elements from the abelian group \(G\) of order \(|G|\) and zero, there are \(k\) non-zero elements per row and column and if \(a = (a_1, \ldots, a_n)\) and \(b = (b_1, \ldots, b_n)\) are any two rows of \(GW(n,k,G)\) then the elements \(a_i b_j^\lambda, i=1, \ldots, n\) give \(\lambda\) copies of \(G\). If \(\lambda\) is a constant for all \(a\) and \(b\) we have a balanced weighing matrix.

Weighing matrices, the special case with \(G\) the cyclic group of order 2 have been studied extensively [10,11,13,19,22]. Their name comes from Yates [26] who gave an application in the accuracy of measurements. Balanced weighing matrices have been studied in connection with combinatorial designs by Mallin and Stanton [14,15,16,21] and Berman [2]. Complex weighing matrices have been studied by Berman [3] and Geramita and Geramita [9].

To illustrate that Berman's generalised weighing matrices and ours are not the same we consider

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
i^2 & i & 0 & 1 \\
i & i^2 & 1 & 0
\end{bmatrix}
\]

which satisfies \(AA^* = 3I\) and is a \(4(2,3,2)\) when \(i^2 = -1\) but is not a generalised weighing matrix by our definition as the product of row 1 and 2 is \((1,1)\) and we need \((1,1,1,1)\).

Notation. Throughout this paper we use \(\mathbb{Z}_q\) for the cyclic group on \(q\) symbols and \(\mathbb{C}_p\) for the elementary abelian group \(\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p\).

For our purposes an \(SBIBD(v,k,\lambda)\) is a matrix with entries 0 and 1 of order \(v\) with \(k\) non-zero rows and columns and inner product between rows of \(\lambda\).

David Glynn [12] has found the only \(GH(n,\mathbb{Z}_2)\) known to the author where \(G\) is
not an abelian group. Consider the multiplication table for $F_6$:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 3 & 4 & 5 & 6 & 1 \\
3 & 4 & 5 & 6 & 1 & 2 \\
4 & 5 & 6 & 1 & 2 & 3 \\
5 & 6 & 1 & 2 & 3 & 4 \\
6 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

Then the circulant matrix with first row
\[
[0 \ 5 \ 1 \ 4 \ 0 \ 1 \ 6 \ 5 \ 0 \ 4 \ 6]
\]
is a generalised weighing matrix $GW(15,5,5,1)$.

2. A FAMILY OF GENERALISED WEIGHTING MATRICES

We first give a more direct construction for a result implicit in the work of Rajkumar. We note that our matrix implies the one of Herman but has an additional property and is obtained quite differently.

Let $\gamma$ be a primitive element of $GF(p^3)$. Let $q | p^3 - 1$ and let $\alpha$ be a generator of $\mathbb{Z}_q$, the cyclic group. Write $\gamma_1 = 1, \gamma_2, \ldots, \gamma_p$ for the elements of $GF(p^3)$ and define $M = (\gamma_i)$ of order $p^3 - 1$ as follows:

\[
\begin{align*}
\gamma_{i+1} & \equiv \gamma_i^k \\
\gamma_{i+j} & \equiv \gamma_i^{\gamma_j} \quad \text{where} \quad \gamma_j \in \mathbb{Z}_q \\
\gamma_{i+j} & \equiv \gamma_i \gamma_j^{\gamma_1} \\
\end{align*}
\]

Example. Let $\gamma$ be a primitive element of $GF(3^2)$ and $\omega$ be a primitive element of $GF(3)$ where $q = 3$. Write $\gamma_1 = 0$, $\gamma_2 = 1$, $\gamma_3 = \gamma$, $\gamma_4 = \gamma + 1$ for the elements of $GF(3^2)$ using $\gamma^2 = \gamma + 1$. Now

\[
M = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & \omega \\
1 & 1 & 0 & \omega^2 \\
1 & \omega & \omega^2 & 1
\end{bmatrix}
\]

We note that $M$ is a $GW(3,3,3,3)$.

Example. Let $\gamma = 3$ be a primitive element of $GF(7)$ and $\omega$ be a primitive element of $GF(7)$ where $q = 7$. Write $\gamma_1 = 1 - 1, 1 = 1, \omega, 7$ for the elements of $GF(7)$. Now
We note $M$ is a $\mathbb{G}(3,7,2)$.

**Theorem 1.** Suppose $p^r$ is a prime power and $q \mid p^{r-1}$. Then there exists a balanced $\mathbb{G}(p^{r-1}, p^r, q)$.

**Proof.** Consider $M$ of order $p^{r-1}$ as above. We show $M$ is the required $\mathbb{G}(p^{r-1}, p^r, q)$. $M$ has the elements $0, 1, \{((p^r-1)/q) + 1\}$ times, and $a, a^2, \ldots, a^{q-1}$ (each $(p^r-1)/q$ times) in each row (column) but the first. So we have the group property with respect to the first row.

We now consider the other rows. We consider $q = p^r - 1$. Suppose $g_i - g_j = s^t$, $g_i - g_j = s^t$ then $g_i - g_j = h^t$. We wish to show that $g_i - g_j = s^t$ cannot arise in any other way. We proceed by reduction ad absurdum. Suppose there exists another entry $g_i - g_j = s^t$ for $i \neq j$. We have that $g_i - g_j = s^t = r^t$ where $r = x = s$. Then $g_i - g_j = r^t = s^t$. But this means there were no other entries. Hence each of the $p^r - 2$ elements $g_i - g_j$ is different. It is not possible for $g_i - g_j$ to be an added entry in the $p^r - 2$ entries are $h^t, h^{2t}, \ldots, h^{(p^r-2)t}$. The 1 comes from $g_i - g_j$.

So we have a generalized $\mathbb{G}(p^r-1, p^r, q)$. The matrix is balanced as the underlying $\mathbb{G}(p^r-1, p^r, q)$.

**Remark.** This construction was first given for $q = 4$ in [19, p.297].

3. SOME GENERALISED HADAMARD MATRICES $\mathbb{G}(p^r, q, p^r)$ and $\mathbb{G}(p^r, (p^r-1), p^r)$

The $\mathbb{G}(p^r, q, p^r)$ was first noted by Drake [8] but we give it here for illustrative purposes.

Let $x$ be a primitive element of $\mathbb{G}(p^r)$. We form

$$X = \{x^i \mid i \equiv 1 \pmod{p^r-1}\}$$

Now the generalized Hadamard matrix on the elementary abelian group in additive form is formed by taking the elements of $X$ modulo a primitive polynomial and adding the zeroth row and column which is the additive identity. This matrix can now be written multiplicatively to obtain $\mathbb{G}(p^r, 2^r, \ldots, 2^r)$. 

| 0 1 1 1 1 1 1 |
| 0 1 w^2 u \cdot u^2 1 |
| 1 0 1 u^3 \cdot u^2 1 |
| 1 1 0 1 u \cdot u^3 1 |
| 1 u w^2 1 0 1 u |
| 1 u \cdot w^2 1 1 0 1 |
| 1 u^2 u \cdot u \cdot u^2 1 0 |

$$M = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & w^2 & u & \cdot & u^2 & 1 \\
1 & 0 & 1 & u^3 & \cdot & u^2 & 1 \\
1 & 1 & 0 & 1 & u & \cdot & u^3 & 1 \\
1 & u & w^2 & 1 & 0 \cdot & 1 & u \\
1 & u & \cdot w^2 & 1 & 1 & 0 \cdot 1 \\
1 & u^2 & u & \cdot u & \cdot u^2 & 1 & 0 \\
\end{bmatrix}$$
For example, let \( x \) be a primitive element of \( \mathbb{F}(S^2) \). We form

\[
X = x \quad x^2 \quad x^3 \quad \ldots \quad x^0
\]

\[
y^0 \quad x \quad y \quad \ldots \quad y^7
\]

\[
x^2 \quad x^3 \quad x^4 \quad \ldots \quad x
\]

then the generalized Hadamard matrix using \( x^2 = x + 1 \) is

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & x+1 & 2x+1 & 2 & 2x & 2x+2 & x+1 \\
0 & 1 & x & x+1 & 2x+1 & 2 & 2x & 2x+2 \\
0 & x+1 & 2x+1 & 2 & 2x & 2x+2 & x+1 & 2 \\
\vdots
\end{array}
\]

or in multiplicative form

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & a & ab & a^2b & a^3 & ab & a & b \\
1 & b & a & ab & a^2b & a^3 & ab & b \\
\vdots
\end{array}
\]

The corresponding matrices, if \( x \) \((-3)\) is a primitive element of \( \mathbb{F}(S) \), are

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
x & x^2 & x^3 & x^4 & 0 & 3 & 4 & 2 & 1 \\
x^6 & x^2 & x^3 & x^4 & 0 & 1 & 3 & 4 & 2 \\
x^3 & x^4 & x^5 & x^6 & 0 & 2 & 1 & 3 & 4 \\
x^2 & x^3 & x^4 & x^5 & 0 & 4 & 2 & 1 & 3 \\
\end{array}
\]

For reference purposes we note the following theorem. A direct proof of (iii), inspired by Majumdar, will appear elsewhere.

**Theorem 2.** (i) Suppose \( p^r \) is a prime power. Then there is a \( \text{GH}(p^r, C_{p^r}) \) where \( C_{p^r} \) is the elementary abelian group.

(ii) Suppose \( p^r \) and \( p^s \) \(-1\) are both prime powers. Then there is a \( \text{GH}(p^r(p^s-1), C_{p^r}) \) where \( C_{p^r} \) is the elementary abelian group.

**Example of construction of \( \text{GH}(15, 2^2\times 5) \)**

\[
\begin{array}{cccc}
d & a & b & ab \\
e & c & a & b \\
\end{array}
\]

has core \( C = e \ a b \)

\[
\begin{array}{cccc}
c & a & b & ab \\
d & e & a & b \\
\end{array}
\]

\[
\begin{array}{cccc}
ab & ab & b & a \\
\end{array}
\]
The generalized Hadamard matrix of order 4:
\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & a & b & ab \\
0 & b & ab & a \\
0 & ab & a & b \\
\end{array}
\]
has core \( K = a \ b \ ab \)

Let \( I, T, T^2 \) of order 3 be a matrix representation of \( e, w, w^2 \) where \( w \) is a cube root of unity, then

\[
W = \begin{bmatrix}
e & e & e \\
e & w & w^2 \\
e & w^2 & w \\
\end{bmatrix}
\]

is a generalized Hadamard matrix of order 3.

Now define

\[
CN = \begin{bmatrix}
e & ab & bc \\
ab & ew & w^2 \\
bc & w^2 & ew \\
\end{bmatrix}
\]

and

\[
D = \begin{bmatrix}
eK & abK & bcK \\
abK & eKT & KT^2 \\
bcK & KT^2 & eKT \\
\end{bmatrix}
\]

and the following is the required matrix:

\[
\begin{bmatrix}
e & e & e & e \\
e & e & e & e \\
e & e & e & e \\
g' & g' & g' & g' \\
g' & g' & g' & g' \\
g' & g' & g' & g' \\
g' & g' & g' & g' \\
\end{bmatrix}
\]

where \( g = [a \ b \ a] \) and \( g' = \begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix} \). Explicitly
4. **Using $G^W(y,k,G)$ to Construct $S_{(1,0)}^B$**

Write $P$ for the matrix with 1 where David Glynn's $G^W(13,9,S_y)$ has zeros and 0 where the $G^W$ is non-zero and $e = (1,1,1,1,1)$. Then, as Glynn observed,

$$DG = \begin{bmatrix} P^T & 1_{1^e}\times e \\ 1_{1^e}\times e^T & G^W(13,9,S_y) \end{bmatrix}$$

is the incidence matrix of the Hughes plane of order $9$.

In general, we can say

**Lemma 3.** Suppose there exists a $G^W(p^2+1,p^2,G)$, $[G] = p(p+1)$. Then forming $DG$ similarly to the above we have the incidence matrix of a tangentially transitive projective plane of order $p^2$.

**Remark.** If $G$ is an "interesting group" then the related projective plane will also be "interesting".

We now give some other constructions using generalised weighing matrices.

**Theorem 4.** Suppose there is a generalised balanced weighing matrix $W = G^W(v,k,S_y)$ with entries, $8^k$, which are $d^{th}$ roots of unity. Suppose the underlying $S_{(1,0)}^B$ has parameters $(v,k,s)$. Then if $G(v-k) = k+1$ there exists a $S_{(1,0)}^B$\[vvd^2, vvd(d^2+1), k(d+1), k, k]\n
and an $S_{(1,0)}^B$\[vvd(d-1)+1, vvd+1, k].\]

**Proof.** Each entry $a_i^k$ of the $G^W(v,k,S)$, is first replaced by $a_i^k G(d,S_y)$ where $G(d,S_y)$ is the generalised Hadamard matrix. Now $W_A$ of order $vvd^2$ with $kd$ ones per row and column is formed by replacing each element, $a_i^k$, by its permutation matrix representation $A_i$ of order $d$. $W_A$ has inner products $s, k, k$.

$W_A$ and $W_C$ are formed by replacing 0 by $a_i^k$ the $d \times d$ zero matrix, and $a_i^k$ by $a_i^k A_i$ and $a_i^k A_i^T$ respectively in $W$, with $e$ the $d \times 1$ matrix of ones. Now $W_A$ has inner
products \( k,1,1/k \), is of size \( \sqrt{d} \times \sqrt{d} \), and has 1 ones per row and \( k \) ones per column.

\[
\begin{bmatrix}
W_A & W_B
\end{bmatrix}
\]

is the required SBIBD.

The matrix \( W_B \) is now obtained by replacing each zero element of \( W \) by \( J_d \) the \( d \times d \) matrix of ones and each non-zero element by \( O_d \). Thus, with \( \mathbf{e} \) the \( 1 \times d \) matrix of ones,

\[
\begin{bmatrix}
\mathbf{e} & \mathbf{0} \\
\mathbf{e}^T & W_B
\end{bmatrix}
\]

is the required SBIBD.

Example. Berman has shown that there is a circulant matrix \( W = W((2^{t+1}-1)/3,2^{t-1}) \), \( t \equiv 3 \) odd, with entries the cube roots of unity \( 1,\omega,\omega^2 \). Since 3 is prime, \( W \) is a balanced \( \text{CG}((2^{t+1}-1)/3,2^{t-1},1) \). We replace each element \( \omega^i \) by

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{bmatrix}
\]

and 0 by \( O_3 \). We form \( M_B \) by replacing \( \mathbf{e} \) by \( \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \) and \( \mathbf{e}^T \) by \( \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \).

\( W_A \) and \( W_C \) are obtained by replacing \( 0 \) by \( O_3 \) and \( \omega^i \) by

\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

respectively.

Since \( W \) is orthogonal, the inner product of any two rows is a multiple \( \lambda \) of

\[
1 + \omega + \omega^2.
\]

Further, since replacing \( 1,\omega,\omega^2 \) by \( \mathbf{e} \) gives the incidence matrix of a \((2^{t+1}-1)/3,2^{t-1},1,2^{t-3}\) difference set, we see \( \lambda = 2^{t-3} \). Now \( W_B \) is of order \( 3(2^{t+1}-1) \) has \( 2^{t-1} \) ones per row and column and has inner products of rows \( 0,2^{t-1} \) or \( 2^{t-3} \). \( W_B \) is of size \( 3(2^{t+1}-1) \times (2^{t+1}-1) \), has \( 2^{t-1} \) ones per row and \( 2^{t-3} \) or \( 2^{t-2} \); \( W_B \) is of size \((2^{t+1}-1)\times (2^{t+1}-1)\), has \( 2^{t-2} \) ones per row and \( 2^{t-2} \) ones per column; further, it has inner products \( 0 \) or \( 2^{t-3} \).

\[
\begin{bmatrix}
W_A & W_B
\end{bmatrix}
\]

is a \( \text{SBIBD} \) \((3(2^{t+1}-1),2^{t+2}-4,2^{t+1}-3,2^{t+1}-1,2^{t-3})\).

We form \( M_B \) by replacing the terms of \( W \) by \( J_d \) and all other elements by \( O_d \).

Since \( M_B \) is based on a \((2^{t+1}-1)/3,(2^{t-1}-1)/3,(2^{t-1}-1)/3\) difference set, it has \( 2^{t-1} \) ones per row and column and inner products \( 2^{t-2}, 1 \) and \( 2^{t-2} \). So with \( \mathbf{e} \) the \( 1 \times (2^{t+1}-1) \) matrix of ones we have
is the incidence matrix of a \( (2^{t+1}, 3, 3^{t+1}, 2^{t+1}) \) SBIBD.

So we have a new proof of a case of a theorem of Rajkumaria.

**Corollary 5.** Let \( t \geq 3 \) be odd. Then there exists an SBIBD with parameters

\[
(2^{t+1}, 3, 3^{t+1}, 2^{t+1})
\]

Example. Berman exhibits a \( N(16, 21) \) with entries which are cube roots of unity.

Since \( d = 3 \), \( v = 21 \), \( k = 16 \) satisfies \( 3(21-16) = 16 - 1 \), the theorem tells us there is an SBIBD \((23, 64, 16)\).

**Corollary 6.** Suppose there is a \( G(p^2+1, p, p^2) \). Then there exists an SBIBD with parameters \((p(p^2-1)+1, p^2, p)\). In particular, an SBIBD \((p(p^2-1)+1, p^2, p)\) exists whenever \( p \) is a prime power.

This family of SBIBDs has recently been found by Becker and Piper (1) and in more general form by Rajkumaria.

**Theorem 7.** Suppose there is a balanced generalized weighing matrix \( G(v, k, \lambda, \epsilon) \). Suppose the underlying SBIBD has parameters \((v, k, \lambda)\). Then if \( v - 1 = (v-2)(d-1) \) there exists an SBIBD \( G(v, k-d(v-1), d(v-1)) \).

**Proof.** Replace each non-zero element by its \( d \times d \) permutation matrix representation and each zero element by the \( d \times d \) matrix of ones. Berman found circulant \( M(2^{t+1}, p, 2^{t+1}) \), \( t \geq 3 \) odd, with entries which are cube roots of unity. Since 3 is a prime, this matrix is a balanced \( M(2^{t+1}, p, 2^{t+1}) \). This satisfies the conditions of the theorem and so we have the family of SBIBDs \( (2^{t+1}, 3, 3^{t+1}, 2^{t+1}) \) which is, of course, well-known.

**Corollary 8.** Suppose there exists a \( G(p^2+1, p^2, p) \). Then there exists an SBIBD \( G(p^2+1, p^2, p) \). This gives the well-known family of SBIBDs \( G(p^2+1, p^2, p) \) when \( p \) is a prime power or in this case we know the \( G(p^2+1, p^2, p) \) exists from Theorem 1.

5. USING GENERALIZED HADAMARD MATRICES

We now give an alternate construction for the SBIBD of Corollary 6.

**Theorem 9.** Suppose there exists a generalized Hadamard matrix \( G(q^p+1, q, q^p) \) where \( G \) is an abelian group. Further, suppose an SBIBD \( G(q^p+1, q, q^p) \) exists with incidence matrix containing \( M_1 \times (q^p+1) \). Then there exists an SBIBD \( G(q^p+1, q^p+1, q^p) \).

**Proof.** Let \( e \) be the \( 1 \times 1 \) matrix of ones and \( J \) the \( t \times t \) matrix of ones. Let \( 0 \) be the zero matrix of size \( xy \). Let \( q \) be a prime of size \( x \), \( y \) and \( z \) respectively, where \( x = p(q^p+q^p) \) and \( y = p^2 \). Let \( A_1, \ldots, A_p \) be the \( pxp \) permutation
matrix representation of \( C_p \). Write \(\text{GH}(A)\) for the \((0,1)\) matrix obtained by replacing each element of \( C_p \) by its appropriate matrix representation. Then \(\text{GH}(A)\) is a

symmetrical group divisible design with parameters

\[
(q^{p+1}(p-1), q^{p+1}(p-1), q^{p+1}(p-1), q^{p+1}(p-1), q^{p+1}(p-1), p).
\]

We write the incidence matrix of the SBIBD \( p(q^{p+1}+1, q^{p+1}, q^{p+1}) \) as

\[
\begin{bmatrix}
N_1 & N_2 \\
N_3 & N_4
\end{bmatrix}
= \begin{bmatrix}
H_1 & X \\
H_3 & Y
\end{bmatrix}
\begin{bmatrix}
N_1 & N_2 \\
N_3 & N_4
\end{bmatrix},
\]

where \( H_1 \) is \((q^{p+1}+1) \times (q^{p+1}+1)\), \( H_2 \) is \((q^{p+1}+1) \times q^{p+1}\), \( H_3 \) is \(q^{p+1} \times (q^{p+1}+1)\) and \( H_4 \) is \(q^{p+1} \times q^{p+1}\). Now form

\[
M = \begin{bmatrix}
N_1 & 0 & a & e_{p}X \\
0 & 0 & c & e_{p}X \\
e_{p}Y & e_{p}Y & 0 & 0 \\
\end{bmatrix}
\]

which is the incidence matrix of the required SBIBD.

We note in passing that

\[
e_{p}Y & \text{GH}(A)
\]

is a pairwise balanced design \( (q^{p+1}(p-1), q^{p+1}(p-1), q^{p+1}(p-1)) \).

In particular, we note that if \( q = 1 \) and \( p \) and \( p - 1 \) are both prime powers, the \(\text{GH}(p^{p+1}, q)\) exists for all positive \( i \), as does the SBIBD \( (p^{p+1}+1, p, 1) \). So an SBIBD \( (p^{p+1}+1, p^{p+1}, p) \) of the right form exists by the theorem. Hence, by induction we have Rajkumaria's theorem as a corollary.

**Corollary 9.** Suppose \( p \) and \( p - 1 \) are prime powers. Then there exists an

SBIBD \( (p^{p+1}+1, p^{p+1}, p) \) for all positive \( i \).

**Example.**
where $c = [1, 1, 1]$ and $f = [1]$ is a $(25, 9, 3)$.

REFERENCES


[22] G.A. Vanstone and R.C. Mullin, A note on existence of weighing matrices \( W_{1}(29,1,3^7) \) and associated combinatorial designs, Utilitas Math. 8 (1975), 371-381.


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