Ordered Partitions and Codes Generated by Circulant Matrices

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We consider the set of ordered partitions of $n$ into $m$ parts acted upon by the cyclic permutation $(12\ldots m)$. The resulting family of orbits $\mathcal{P}(n,m)$ is shown to have cardinality $p(n,m) = \frac{1}{m} \sum_{d|m} \varphi(d) \binom{n/d}{m-1}$, where $\varphi$ is Euler's $\phi$-function. $\mathcal{P}(n,m)$ is shown to be $m$-isomorphic to the family of orbits $\mathcal{P}(n,m)$ of the set of all $m$-subsets of an $n$-set acted upon by the cyclic permutation $(12\ldots m)$. This isomorphism yields an efficient method for determining the complete weight enumerator of any code generated by a circulant matrix.

1. INTRODUCTION

An ordered partition (or composition, cf. [2] or $m$-composition, cf. [1]) of $n$ into $m$ parts is an ordered $m$-tuple $\alpha = (k_1, k_2, \ldots, k_m)$, where the $k_i$ are positive integers and $k_1 + k_2 + \cdots + k_m = n$. In this paper we consider the set $\mathcal{P}(n,m)$ of all ordered partitions of $n$ into $m$ parts acted upon by the cyclic permutation

$$\theta = (12\ldots m).$$

The action of group $G$ generated by $\theta$ is defined by

$$\theta\kappa = (k_{1\theta}, k_{2\theta}, \ldots, k_{m\theta}).$$
and we write \( \mathcal{P}(n, m) \) for the set of orbits of \( G \) under this action. The cardinalities of \( \mathcal{P}(n, m) \) and \( \mathcal{Q}(n, m) \) will be denoted by \( p(n, m) \) and \( q(n, m) \), respectively. Writing \( \tilde{p}_d(n, m) \) for the number of orbits in \( \mathcal{P}(n, m) \) having exactly \( d \)-elements, we derive in Section 3 the identities

\[
\tilde{p}_d(n, m) = \frac{1}{n} \sum_{d \mid m} \mu(d) \left\lfloor \frac{n}{d} \right\rfloor \frac{m}{d} \tag{1.1}
\]

and

\[
\tilde{q}_d(n, m) = \frac{1}{n} \sum_{d \mid m} \phi(d) \left\lfloor \frac{n}{d} \right\rfloor \frac{m}{d}, \tag{1.2}
\]

where \( \mu \) is the Möbius function, \( \phi \) is Euler's \( \phi \)-function, and \( (a, b) \) is defined to be zero unless \( a \) is a divisor of both \( n \) and \( m \).

The initial reason for our interest in the set \( \mathcal{P}(n, m) \) is due to the fundamental relationship between \( \mathcal{P}(n, m) \) and the set of all \( m \)-subsets of a given \( n \)-set. Write \( S \) for the set of integers \( \{1, 2, \ldots, n\} \) and \( \mathcal{C}(n, m) \) for the set of all \( m \)-subsets of \( S \). Let \( H \) be the cyclic group generated by the permutation

\[
\psi = (1 \ 2 \ \cdots \ n).
\]

For \( l = \{a_1, a_2, \ldots, a_m\} \), any element of \( \mathcal{C}(n, m) \), we define the action of \( H \) on \( \mathcal{C}(n, m) \) by

\[
\psi l = \{a_1\psi, a_2\psi, \ldots, a_m\psi\}, \tag{1.3}
\]

i.e.,

\[
a_i\psi \equiv a_i \pmod{n}.
\]

The set \( \mathcal{Q}(n, m) \) of orbits of \( H \) is shown in Section 2 to be set-isomorphic to \( \mathcal{P}(n, m) \), and the properties of the isomorphism are studied in some detail.

The isomorphism between \( \mathcal{Q}(n, m) \) and \( \mathcal{P}(n, m) \) yields an efficient method for determining the complete weight enumerator of any code generated by the row vectors of a circulant matrix or a matrix of the form \([IW]\), where \( I \) is the \( n \times n \) identity matrix and \( W \) is an \( n \times n \) circulant matrix. This application is discussed in Section 4.

2. The Relationship between Ordered Partitions and \( m \)-Sets

The purpose of this section is to establish the fundamental relationship between the two sets \( \mathcal{P}(n, m) \) and \( \mathcal{C}(n, m) \). We will denote the cardinalities of \( \mathcal{Q}(n, m) \) and \( \mathcal{C}(n, m) \) by \( q(n, m) \) and \( c(n, m) \), respectively. The number of orbits in \( \mathcal{C}(n, m) \) with \( d \) elements will be denoted by \( \tilde{c}_d(n, m) \).
Each $m$-subset of $S$ has a natural ordering. Let $l = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_m$. Associated with $l$ we have the ordered partition of $n$ into $m$ parts

$$\alpha(l) = (d_1, d_2, \ldots, d_m)$$

(2.1)

defined by

$$d_i = \alpha_{i+1} - \alpha_i \quad \text{for} \quad i = 1, \ldots, m - 1,$$

$$d_m = n - \alpha_m - \alpha_1.$$  

Also, with each ordered partition $x = (k_1, k_2, \ldots, k_m)$ we associate the $m$-set

$$l(x) = \{1, 1 + k_1, \ldots, 1 + k_1 + k_2 + \cdots + k_{m-1}\}. \quad (2.2)$$

We prove next that (2.1) and (2.2) yield a bijection between the sets $\mathcal{A}(n, m)$ and $\mathcal{B}(n, m)$.

**Lemma 2.1.** The ordered partitions associated with a class in $\mathcal{B}(n, m)$ are contained in a class in $\mathcal{A}(n, m)$.

**Proof.** Let $l = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_m \leq n$, and let $\alpha(l) = (d_1, d_2, \ldots, d_m)$ be defined by (2.1). Then

$$\psi^l = \{\alpha_1 + k, \alpha_2 + k, \ldots, \alpha_m + k\},$$

where the elements are reduced modulo $n$. In natural order

$$\psi^l = \{\alpha_1 + k, \alpha_{i+1} + k, \ldots, \alpha_m + k, \alpha_{i} + k, \ldots, \alpha_{i-1} + k\},$$

for some integer $i$. Hence the ordered partition associated with $\psi^l$ is

$$\alpha(\psi^l) = (d_i, \ldots, d_{i-1}, \alpha_1 - \alpha_m, d_i, \ldots, d_{i-1}, n - \alpha_{i-1} - k + \alpha_i + k).$$

But

$$\alpha_1 - \alpha_m = d_m \quad (\text{mod } n)$$

and

$$n - \alpha_{i-1} - k + \alpha_i + k = d_{i-1} \quad (\text{mod } n),$$

and so

$$\alpha(\psi^l) = \theta^{i-1}\alpha(l), \quad (2.3)$$

which proves the assertion of the lemma.
Lemma 2.2. The m-sets associated with a class in $\mathcal{A}(n, m)$ are contained in a class in $\mathcal{B}(n, m)$. In particular

$$l(\theta^i) = \psi^i l(\alpha)$$

for $i = 0, 1, \ldots, m - 1$, where $b_i = k_{i+1} + k_{i+2} + \cdots + k_m$.

Proof. By definition

$$\psi^i l(\alpha) = \{1 + b_1, 1 + b_2 + k_1, \ldots, 1 + b_i + k_1 + \cdots + k_{m-1}\}.$$ 

Since

$$1 + b_i + k_1 + \cdots + k_i \equiv 1 \pmod{n}$$

we have in natural order

$$\psi^i l(\alpha) = \{1, 1 + k_{i+1}, \ldots, 1 + k_{i+1} + \cdots + k_{m-1}, 1 + k_{i+2} + \cdots + k_m, \ldots, 1 + k_{i+1} + \cdots + k_m + k_1, \ldots, 1 + k_{i+1} + \cdots + k_{m-1}\}$$

$$= l(\theta^i\alpha).$$

Theorem 2.1. Define $f: \mathcal{A}(n, m) \to \mathcal{B}(n, m)$ by

$$f(\alpha) = [l(\alpha)]$$

and define

$$g: \mathcal{B}(n, m) \to \mathcal{A}(n, m)$$

by

$$g(\alpha) = [\alpha(f)],$$

where the representative $l$ contains 1.

Then $f$ and $g$ are well defined and $f \circ g = 1$, $g \circ f = 1$.

Proof. $f$ is well defined by Lemma 2.2 and $g$ is well defined by Lemma 2.1; hence it suffices to prove that $f$ and $g$ are mutual inverses.

Let $I = \{a_1, a_2, \ldots, a_m\}$ and write $[I]$ for the corresponding class in $\mathcal{B}(n, m)$. Then for $\alpha(l) = (a_1, a_2, \ldots, a_m)$ defined by (2.1) we have that

$$l(\alpha(l)) = \psi^{l\alpha(l)}l;$$

hence $[l(\alpha(l))] = [I]$ and so $f \circ g = 1$.

On the other hand, let $\alpha = (k_1, k_2, \ldots, k_m)$. Then by (2.2)

$$l(\alpha) = \{1, 1 + k_1, \ldots, 1 + k_1 + \cdots + k_{m-1}\}.$$
and by (2.1)
\[ \alpha(l(\alpha)) = (d_1, d_2, \ldots, d_m), \]
where
\[ d_1 = 1 + k_1 - 1 - k_1, \quad d_2 = 1 + k_1 + k_2 - 1 - k_1 = k_2, \ldots, \quad d_{m-1} = k_{m-1} \]
and
\[ d_m = n - (1 + k_1 + \cdots + k_{m-1}) + 1 = k_m. \]
Hence
\[ \alpha(l(\alpha)) = \alpha, \]
and so \( \{\alpha(l(\alpha))\} = \{\alpha\} \), which proves that \( g \circ f = 1 \). This completes the proof of the theorem.

An immediate consequence of Theorem 2.1 is
\[ \bar{p}(n, m) = \bar{c}(n, m). \tag{2.7} \]

The next theorem shows that the bijection \( f \) preserves, in a sense, the class size.

**Theorem 2.2.** Let \( f \) be the mapping defined by Eq. (2.5) and suppose \( k \) is a divisor of \( m \). If \( \{\alpha\} \in \mathcal{A}(n, m) \) is a class containing \( m/k \) elements then the class \( f(\{\alpha\}) \) contains \( n/k \) elements.

**Proof.** Suppose \( \{\alpha\} \) contains \( m/k \) elements. Then
\[ \alpha = (k_1, \ldots, k_d, k_1, \ldots, k_d, \ldots, k_1, \ldots, k_d), \]
where \( d = m/k \) and each \( d \)-tuple \((k_1, \ldots, k_d)\) is an ordered partition of \( n/k \) into \( m/k \) parts whose class in \( \mathcal{A}(n/k, m/k) \) contains exactly \( m/k \) elements.

Write \( h = n/k \). Then
\[ l(\alpha) = \{1, 1 + k_1, \ldots, 1 + k_2, \ldots, k_{d-1}, 1 + h, 1 + h + k_1, \ldots, \}
\[ 1 + (k - 1)h + k_1 + \cdots + k_{d-1}. \]
Hence \( \psi^h(l(\alpha)) = l(\alpha) \), from which it follows that
\[ f(\{\alpha\}) = \{l(\alpha)\} \text{ contains } h = n/k \text{ distinct elements.} \]

**Corollary.** The following identity holds for \( k \mid (m, n), \)
\[ \bar{c}_{m/k}(n, m) = \bar{p}_{m/k}(n, m). \]
To each \( m \)-subset \( l \) of \( S \) there corresponds the \((n - m)\)-subset \( S - l \). This correspondence defines a natural bijection between \( \mathcal{E}(n, m) \) and \( \mathcal{E}(n, n - m) \). Moreover since
\[
S - \psi l = \psi S - \psi l = \psi(S - l)
\]
the mapping
\[
t : \mathcal{E}(n, m) \to \mathcal{E}(n, n - m),
\]
defined by
\[
t[l] = [S - l],
\]
is well defined and is a bijection.

Incorporating the results of Theorem 2.1 we have the commutative diagram
\[
\begin{align*}
\mathcal{E}(n, m) & \xrightarrow{t} \mathcal{E}(n, n - m) \\
\mathcal{P}(n, m) & \xrightarrow{g \circ f} \mathcal{P}(n, n - m)
\end{align*}
\]
where \( g \circ f : [\alpha] \to [\alpha(S - l(\alpha))] \).

Since \( f, t, \) and \( g \) are bijections we can conclude that \( g \circ t \circ f \) is also. Suppose next that \([l]\) is a class in \( \mathcal{E}(n, m) \) with \( n/k \) elements; then if \( h = n/k \) we have
\[
\psi hl = l
\]
and consequently
\[
S - l = S - \psi hl = \psi(S - l).
\]

This shows that classes with \( n/k \) elements in \( \mathcal{E}(n, m) \) are in one-one correspondence with classes having \( n/k \) elements in \( \mathcal{E}(n, n - m) \).

Hence we have the following theorem.

**Theorem 2.3.** The mapping \( g \circ t \circ f \) defined in (2.9) is a bijection between \( \mathcal{P}(n, m) \) and \( \mathcal{P}(n, n - m) \) which maps classes containing \( m/k \) elements to classes containing \((n - m)/k \) elements.

**Corollary.**
1. \( \hat{c}(n, m) = \hat{c}(n, n - m) \),
2. \( \hat{p}(n, m) = \hat{p}(n, n - m) \),
3. \( \hat{p}_{m/k}(n, m) = \hat{p}_{(n-m)/k}(n, n - m) \).
3. The Cardinality of $\mathcal{P}(n, m)$

In this section we derive (1.1) and (1.2). Since $p(n, m)$ can be interpreted as the number of ways of inserting $m - 1$ commas into $n - 1$ places [2] we have

$$p(n, m) = \binom{n - 1}{m - 1} = \frac{m}{n} \binom{n}{m}. \quad (3.1)$$

Also, the cardinality of each orbit is a divisor of $m$. Hence we immediately have the equations

$$\frac{m}{n} \binom{n}{m} = p(n, m) = \sum_{d|m} \bar{p}_d(n, m) \quad (3.2)$$

and

$$\bar{p}(n, m) = \sum_{d|m} \bar{p}_d(n, m). \quad (3.3)$$

Perhaps the most elegant way to obtain (1.1) is to observe that $p((n/m)k, k)$ is defined for all positive integers $k$, if we let $p((n/m)k, k) = 0$ whenever $(n/m)k$ is not an integer; i.e., we define $\binom{n}{x} = 0$ if $nk/m$ is not an integer. Moreover, $\bar{p}_d(n, m)$ is defined for all positive integers $d$, being equal to $0$ whenever $d$ is not a divisor of $(n, m)$, the greatest common divisor of $n$ and $m$. With these observations, we may invert (3.2) to obtain

$$m\bar{p}_m(n, m) = \sum_{d|m} \mu(d) \bar{p}\left(\frac{n}{m}, \frac{m}{d}, \frac{m}{d}\right). \quad (3.4)$$

Equation (1.1) is a trivial consequence of (3.1) and (3.4).

To obtain (1.2) we recall that $G$, the cyclic group of order $m$, acts on the set $\mathcal{P}(n, m)$ of all ordered partitions of $n$ into $m$ parts. Let $\lambda(g)$ denote the number of elements of $\mathcal{P}(n, m)$ fixed by the permutation $g \in G$. If $g = e$, the identity element, then

$$\lambda(g) = \binom{n - 1}{m - 1}$$

since $e$ fixes every ordered partition. If $g$ consists of $d$-cycles then $g$ fixes only those ordered partitions which are repeated copies of ordered partitions of $n/d$ into $m/d$ parts. For example, $(1, 3, 2, 1, 3, 2, 1, 3, 2)$ is fixed by $(147)(258)(369) = (123456789)^3$. But the number of permutations of $G$ consisting of $d$-cycles is $\phi(d)$. Hence by Burnside's lemma

$$\bar{p}(n, m) = \frac{1}{m} \sum_{d|m} \phi(d) \left(\frac{n}{d} - 1\right) = \frac{1}{n} \sum_{d|m} \phi(d) \left(\frac{n}{d}\right)$$
As an example suppose that \( n = 24 \) and \( m = 4 \). Then

\[
\bar{p}(24, 4) = \frac{1}{24} \left[ \phi(1) \binom{24}{4} + \phi(2) \binom{12}{2} + \phi(4) \binom{6}{1} \right]
\]

\[
= \frac{1}{24} \left[ \binom{24}{4} + \binom{12}{2} + 2 \binom{6}{1} \right] = 446.
\]

The following corollaries may serve as further illustrations.1

**Corollary 1.** If \( n \) and \( m \) are relatively prime then

\[
\bar{p}(n, m) = \bar{p}_n(n, m) = \frac{1}{n} \binom{n}{m}.
\]

**Corollary 2.** If \( (n, m) = q \) is a prime then

\[
\bar{p}(n, m) = \frac{1}{n} \binom{n}{m} + \frac{n - q}{n} \binom{n/q}{m/q}.
\]

**Corollary 3.**

\[
\bar{p}(n, 3) = \frac{1}{n} \binom{n}{3} \quad \text{if} \quad (n, 3) = 1
\]

\[
= \frac{1}{n} \binom{n}{3} + \frac{2}{3} \quad \text{if} \quad (n, 3) = 3,
\]

\[
\bar{p}(n, 4) = \frac{1}{n} \binom{n}{4} \quad \text{if} \quad (n, 4) = 1
\]

\[
= \frac{1}{n} \binom{n}{4} + \frac{n}{8} - \frac{1}{4} \quad \text{if} \quad (n, 4) = 2
\]

\[
= \frac{1}{n} \binom{n}{4} + \frac{n}{8} + \frac{1}{4} \quad \text{if} \quad (n, 4) = 4.
\]

4. An Application

Let \( \mathcal{C} \) be a linear code generated by the row vectors of a matrix \([I:W]\), where \( I \) is \( n \times n \) identity matrix and \( W \) is an \( n \times n \) circulant matrix with entries in a finite field \( GF(q) \). Such codes have the property that they have the same weight enumerators as their duals [4] and hence share many of the

1 Added in proof. The total number of ordered partition classes of \( n \) is \( \bar{p}(n) = \sum_{m|n} \bar{p}(n, m) = (1/n) \sum_{m|n} \phi(d)2^{n/d} - 1 \). We are grateful to Professor G. Baur of the Technical University, Vienna, for this observation.
properties of self-dual codes. The design properties of linear codes and their subcodes of constant weight are closely related to their weight enumerators [3]. In general the problem of determining the weight enumerator (WE) of a code, or better still the complete weight enumerator (CWE), involves the determination of the WE or CWE of each of the \(q^n\) codewords. If \(W\) is circulant and \(W_i\) denotes the \(i\)th row of \(W\) then the linear combination

\[ W_{i_1} + W_{i_2} + \cdots + W_{i_m} \]

has the same CWE as

\[ W_{i_1+k} + W_{i_2+k} + \cdots + W_{i_m+k} \]

for any integer \(k\), where the subscripts are reduced modulo \(n\). Hence the codewords of \(\mathcal{C}\) can be grouped into classes in which elements are "linear shifts" of one another. For given \(m\) the family of classes is in obvious correspondence with \(\mathcal{H}(n, m)\). Hence the problem of determining the CWE of \(\mathcal{C}\) reduces to two problems:

1. Finding a complete system of coset representatives of \(\mathcal{H}(n, m)\) for \(m = 1, \ldots, n\).

2. Determining the CWEs of the linear combinations corresponding to the coset representatives.

The problem of finding a complete system of coset representatives is very easy for \(\mathcal{H}(n, m)\), where such a system occurs in lexicographical order among the set of all ordered partitions of \(n\) into \(m\) parts with the first entry at most the integer part of \(n/m\). An ordered partition in this class is a suitable representative provided that it is lexicographically less than any ordered partition in its orbit. An efficient computer algorithm exists to determine the complete system of representatives for \(\mathcal{H}(n, m)\).

We may note that in the case of binary codes Theorem 2.3 allows us to reduce the calculation time by a further factor of 2.

References


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