A COMPUTER LISTING OF HADAMARD MATRICES

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ABSTRACT.

A computer has been used to list all known Hadamard matrices of order less than 40,000. If an Hadamard matrix is not known of order 4q (q odd) then the smallest t so that there is an Hadamard matrix of order 2^t q is given.

Hadamard matrices are not yet known for orders 268, 412, 428.

INTRODUCTION.

An Hadamard matrix of order n has every entry +1 or -1 and its distinct row vectors are orthogonal.

These were discussed by Sylvester [16] in 1867 and Hadamard [8] conjectured in 1893 that they exist for orders 1, 2 and 4t, for every natural number t. Hadamard proved that any complex n x n matrix \( A = (a_{ij}) \) with entries in the unit disc satisfies

\[
(\det A)^2 \leq \prod_{i=1}^{n} \prod_{j=1}^{n} |a_{ij}|^2,
\]

and Hadamard matrices satisfy the equality of this inequality.

In 1933 Paley [10] produced a list showing that Hadamard matrices of orders 92, 116, 156, 172, 184 and 188 where the only unsolved cases of order \( \leq 200 \).

The existence of the matrix of order 172 was settled by Williamson [27] in 1944.

This list induced L.D. Baumert, S.W. Golomb and Marshall Hall Jr to use sophisticated and exciting computer techniques with
Williamson's method to find the Hadamard matrices of orders 92 and 184 (in 1962 [2]), 116 (in 1966 [1]) and 156 (in 1965 [3]).

The case for 188 has been settled by R.J. Turyn using a technique of Goethals and Seidel [7] which generalized that of Williamson.

These results, and those of E. Spence, J. Cooper, J.S. Wallis and A.L. Whiteman have largely given Hadamard matrices of "low" order.

Four matrices $W_1$, $W_2$, $W_3$, $W_4$ of order $w$ with entries +1 or -1 which satisfy

$$W_i W_j^T = W_j W_i^T \quad i, j \in \{1, 2, 3, 4\},$$

$$\sum_{i=1}^{4} W_i W_i^T = 4w1_w,$$

are called Williamson matrices.

A square matrix $A$ of order $n$ with entries from the set of commuting variables $\{0, x_1, x_2, ..., x_8\}$ will be called an orthogonal design of order $n$ and type $(u_1, u_2, ... , u_8)$ if $x_i$ occurs $u_i$ times in each row and column of $A$ and if the rows of $A$ are formally orthogonal.

Then we can express the highly important result of Baumert and Hall (1965 [3]) as

**BAUMERT-HALL THEOREM.** If there is an orthogonal design of order $4t$ and type $(t, t, t, t)$ and Williamson matrices of order $w$, then there exists an Hadamard matrix of order $4wt$.

Orthogonal designs, introduced by Geramita, Geramita and Wallis [6], were used by Wallis [22] to prove

**THEOREM.** Let $q$ be any odd natural number. Then there exists an integer $t = \lceil 2 \log_2(q-1) \rceil$ such that there exists an Hadamard matrix of order $2^t q$ for every natural number $w \geq t$. 
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If an odd number q does not appear in the list it means that there is an Hadamard matrix of order 4q. When q appears in the list the number t next to q indicates there is an Hadamard matrix of order 2tq and no smaller power of 2.
If an odd number \( q \) does not appear in the list it means there is an Hadamard matrix of order \( 4q \). When \( q \) appears in the list the number \( t \) next to \( q \) indicates there is an Hadamard matrix of order \( 2^tq \) and no smaller power of 2.
This result can be used to prove

**Theorem.** Let \( q \) be any odd natural number. Then there exist an integer \( t \) such that there exists a symmetric Hadamard matrix with constant diagonal of order \( 2^{2s}q^2 \) for every natural number \( s \geq t \).

The list.

As mentioned previously, Bailey [10] constructed a list of orders \( \leq 200 \) for which Hadamard matrices were known in 1933. Various other lists appeared (see, for example, Florek [5], Raghavachrao [12], Wallis [23]).

The current listing, which is available for \( q < 10,000 \) although we only give \( q < 2,000 \) in this note, has no entry after \( q \) if an Hadamard matrix of order \( 4q \) is known. If no Hadamard matrix of order \( 4q \) is known then the smallest \( t \) such that an Hadamard matrix of order \( 2^t q \) is known is given. We note that Wallis' theorem gives an upper bound on \( t \) and often Hadamard matrices are known for smaller powers of \( 2 \) (but greater than 2) than that theorem indicates.

The computer tape on which these results are stored in order to be easily updated also has an indication of whether a skew-Hadamard matrix is known.

**Acknowledgements.**

I wish to thank Emma Lehmer and D.H. Lehmer who suggested to me the idea of computerizing this list. The largest part of the programming was done by Dr. Ian S. Williams while a student in the Research School of Physical Sciences at A.N.U. on the School's BRC-10 system. The updating and printing programmes were written by Mr. N. Wormald.

**References.**


(6) Anthony V. Geramita, Joan Murphy Geramita and Jennifer Seberry Wallis, "Orthogonal designs", Linear and Multilinear Algebra, 3 (1975/76), 281-306.


(16) J.J. Sylvester, "Thoughts on inverse orthogonal matrices, simultaneous sign successions, and tessellated payments in two or more colours, with applications to Newton's Rule, ornamental title work, and the theory of numbers", Phil. Mag. No. 4, 34 (1867), 461-475.


