Orthogonal Designs

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INTRODUCTION

Definition. An orthogonal design of order $n$ and type $(s_1, \ldots, s_t)$ ($s_i > 0$) on the commuting variables $x_1, \ldots, x_t$ is an $n \times n$ matrix $A$ with entries from \{0, $\pm x_1, \ldots, \pm x_t$\} such that

$$AA^t = \left( \sum_{i=1}^{t} s_i x_i^2 \right) I_n.$$  \hspace{1cm} (§)

Alternatively, the rows of $A$ are formally orthogonal and each row has precisely $s_i$ entries of the type $\pm x_i$.

Remark. $A$ may be considered as a matrix with entries in the field of quotients of the integral domain $Z[x_1, \ldots, x_t]$. Thus we obtain

$$A^t A = \left( \sum_{i=1}^{t} s_i x_i^2 \right) I_n$$

and so our alternative description above applies equally well to the columns of $A$.

Orthogonal designs of special type have been extensively studied, and it is the existence of these special types that has motivated our study of the general problem of the existence of orthogonal designs.

This paper is organized in the following way. In the first section we give some easily obtainable necessary conditions for the existence of orthogonal designs of various order and type. In Section 2 we briefly survey the examples of such designs that we have found in the literature. In the third section we describe several methods for constructing orthogonal designs. In the fourth

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section we obtain some sharper necessary conditions for the existence of orthogonal designs. In the fifth section we apply the results obtained to calculate designs of small order and also improve some of the results of [14]. We conclude this paper with a collection of open questions and conjectures.

1. SOME EXISTENCE THEOREMS

In this section we shall describe some necessary conditions for the existence of an orthogonal design. The most fundamental restriction concerns the maximum number of variables that can occur in an orthogonal design of order $n$. It is an immediate consequence of a theorem of Radon [8].

**Definition** Let $n = 2^a \cdot b$, $b$ odd, and write $a = 4c + d$ where $0 \leq d < 4$. Radon’s function is the arithmetic function

$$
\rho(n) = 8c + 2d.
$$

Note $\rho(n) = n$ iff $n = 1, 2, 4, 8$. $\rho(16) = 9$, and in general $\rho(2^a \cdot b) = \rho(2^a)$ if $b$ is odd.

**Theorem** Let $A$ be an orthogonal design of order $n$ and type $(s_1, \ldots, s_l)$ on the variables $x_1, \ldots, x_l$. Then $l \leq \rho(n)$.

**Proof** Write $A = A_1x_1 + \ldots + A_lx_l$ where $A_i$ are $(0, 1, -1)$ matrices of order $n$. Now

$$
AA' = \left( \sum_{i=1}^{l} s_i x_i^2 \right) I_n \Leftrightarrow A_i A'_i = s_i I_n, \quad i = 1, 2, \ldots, l,
$$

and $A_i A'_j + A_j A'_i = 0$ for $i \neq j$. If we replace $A_i$ by the real matrix $(1/\sqrt{s_i})A_i = B_i$, then the $B_i$ are real orthogonal matrices satisfying $B_i B'_i + B_j B'_j = 0$ for $i \neq j$, and Radon has shown that there do not exist more than $\rho(n)$ such real matrices. This completes the proof.

So, with the initial limitations of this theorem in mind we propose as a problem finding all orthogonal designs of order $n$. We note that this theorem gives only a weak necessary condition and that conditions for sufficiency are far deeper.

To consider further the question of necessity we introduce weighing matrices.

**Definition** A weighing matrix of weight $k$ and order $n$, is a $(0, 1, -1)$ matrix, $A$, of order $n$ satisfying

$$
AA' = k I_n.
$$

We shall denote such a matrix by $W(n, k)$. Thus, our definition of an orthogonal design could be reformulated in terms of the existence of certain weighing matrices satisfying the equation $XY' + YX' = 0$ in pairs.
If \( n \) is odd then \( \rho(n) = 1 \) and Theorem 1 shows we can only find designs on one variable. However, as observed above, finding an orthogonal design of order \( n \) and type \((s)\) is equivalent to finding a \( W(n, s) \).

If \( n \) is odd the type of a design is severely restricted as we have:

**Proposition 2** If \( n \) is odd and a \( W(n, s) \) exists, then \( s \) is a square.

**Proof** If \( A \) is a \( W(n, s) \) i.e. \( AA^t = sI_n \), then the proposition follows immediately from the observation that \( \det A \) must be an integer.

We note that a \( W(5, 4) \) does not exist but a \( W(13, 9) \) does. (We shall exhibit the \( W(13, 9) \) later.) Thus, Proposition 2 is a weak necessary condition. In fact, the existence question for weighing matrices of odd order has some very interesting consequences. We shall discuss this further in Section 4.

If \( n \equiv 2(mod \ 4) \) we have \( \rho(n) = 2 \) and so we need only consider orthogonal designs on \( \leq 2 \) variables. The following proposition is relevant.

**Proposition 3** [9] If \( n \equiv 2(mod \ 4) \) and \( A \) is a \( W(n, k) \) then \( k \) is a sum of two squares.

**Corollary** If \( n \equiv 2(mod \ 4) \) and \( A \) is an orthogonal design of order \( n \) and type \((s_1, s_2)\) then \( s_1, s_2 \) and \( s_1 + s_2 \) must each be the sum of two squares.

**Proof** If \( A \) is an orthogonal design of order \( n \) and type \((s_1, s_2)\) on the variables \( x_1, x_2 \) then we have noted that this implies the existence of a \( W(n, s_1) \) and \( W(n, s_2) \), thus by Proposition 3, \( s_1 \) and \( s_2 \) are each a sum of two squares. Now, if we set \( x_1 = x_2 = 1 \) in \( A \) then \( A \) becomes a \( W(n, s_1 + s_2) \), and so again, by Proposition 3, \( s_1 + s_2 \) is a sum of two squares.

Finally if \( n \equiv 0(mod \ 4) \) then \( \rho(n) \geq 4 \). It is in these orders that questions of necessary conditions for the existence of orthogonal designs become more delicate. If \( n = 2^t \) for any integer \( t \), we have been unable to find any condition (apart from Theorem 1) which precludes the existence of an orthogonal design of order \( n \). If \( n = 4t \), where \( t \) is odd, we have had some success in finding some necessary conditions which assert the nonexistence of designs of order \( n \) (see Section 4). However, if \( n = 8t \), \( t \) odd, we have been unable to eliminate any \( 4 \)-tuple \((s_1, \ldots, s_4)\), \((where \ l \leq 8 \ and \ \Sigma_{i=1}^4 s_i \leq n)\) as the type of an orthogonal design of order \( n \).

2. KNOWN CLASSES OF ORTHOGONAL DESIGNS

**Baumert-Hall arrays**

Let \( n = 4t \), \( t \) odd, then \( \rho(n) = 4 \). The Baumert-Hall arrays are orthogonal designs of order \( n \) and type \((t, t, t, t)\). These exist for a large number of odd \( t \) and it has been conjectured that they exist for all odd \( t \).

For a discussion of these arrays see [16; Part 4, Chapter VII].
Designs of type \((1, 1, \ldots, 1)\)

Let \(n\) be any integer. In [2] it was shown that there is an orthogonal design of order \(n\) and type \((1, 1, \ldots, 1)\) on the variables \(x_1, \ldots, x_{\rho(n)}\).

Plotkin's array

Let \(n = 8t, t\) odd, then \(\rho(n) = 8\). A Plotkin array is an orthogonal design of order \(n\) and type \((t, t, t, t, t, t, t, t)\). If \(t = 1\) such an array is a classical one derived from the multiplication of the Cayley numbers. Plotkin has exhibited such an array for \(t = 3\) [7]. These appear to be the only such arrays (for \(t\) odd) in the literature. It would be interesting to have more.

In his paper [7] Plotkin also proves the following about orthogonal designs: in our terminology, he shows that if \(n\) is the order of an Hadamard matrix then

a) there is an orthogonal design of type \(\binom{n}{2, 2}\) on the variables \(x_1, x_2\).

b) There is an orthogonal design of order \(2n\) and type \(\binom{n}{2, 2, 2}\), and

c) an orthogonal design of order \(4n\) and type \(\binom{n}{2, 2, 2, 2}\).

Plotkin makes the very strong conjecture that every Hadamard matrix of order \(8n\) can be obtained from specializing some orthogonal design of order \(8n\) and type \((n, n, n, n, n, n, n, n)\), i.e. every Hadamard matrix of order \(8n\) may be obtained from an orthogonal design of the order and type above by setting the variables all equal to 1.

Designs on one variable

As we have noted these are weighing matrices. It has been conjectured [14] that weighing matrices \(W(n, k)\) exist for every \(k \leq n\) when \(n \equiv 0 \pmod{4}\). (This includes the Hadamard conjecture: that for \(n \equiv 0 \pmod{4}\) there exists a \(W(n, n)\) for all integers \(n \geq 4\).) This conjecture has been verified for \(n = 2^t\) [4] and for \(n \in \{12, 20, 24, 28, 32, 40\}\) [14].

3. PROLIFERATING ORTHOGONAL DESIGNS

**Lemma 4** If \(A\) is an orthogonal design of order \(n\) and type \((s_1, \ldots, s_l)\) on the variables \(x_1, \ldots, x_l\) then there is an orthogonal design of order \(n\) and type \((s_1, \ldots, s_l, s)\) on the \(l - 1\) variables \(x_1, \ldots, \hat{x}_j, \ldots, x_l\).

**Corollary 1** For any integer \(n\), an orthogonal design of order \(n\) and type \((s_1, \ldots, s_l)\) exists if \(\sum_{i=1}^{l} s_i \leq \rho(n)\) and so if \(n = 2, 4, 8\) orthogonal designs of order \(n\) and any type exist.

**Proof** The proof follows by using the designs of type \((1, 1, \ldots, 1)\) in the previous section.
COROLLARY 2  If \( n = 4t \) and there exists a Baumert-Hall array of order \( 4t \), then there exist orthogonal designs of order \( n \) and type

\[
(t), (2t), (3t), (4t), (t, t, 2t), (t, t), (t, 3t)
\]

\[
(t, t, t), (t, t), (2t, 2t)
\]

Proof  These designs result from setting the variables in the Baumert-Hall array equal to each other or to 0.

LEMMA 5  If \( A \) is an orthogonal design of order \( n \) and type \((s_1, \ldots, s_t)\) on \( x_1, \ldots, x_t \), then there exists an orthogonal design of order \( mn \) and of type \((s_1, \ldots, s_t)\) on \( x_1, \ldots, x_t \) for any integer \( m \geq 1 \).

PROPOSITION 6  Let \( A_1, \ldots, A_s \) be matrices of order \( m \) satisfying

i) \( A_iA_i^t = a_iI_m \quad i = 1, \ldots, s \)

ii) \( A_iA_j^t + A_jA_i^t = 0 \quad (i \neq j) \)

and let \( N_1, \ldots, N_i \) be matrices of order \( n \) satisfying

iii) \( N_iN_i^t = b_iJ_n \quad i = 1, \ldots, l \)

iv) \( N_iN_j^t = N_jN_i^t \quad i \neq j \).

Then if \( B_1 = A_1 \otimes N_1, \ldots, B_s = A_s \otimes N_s(k_1 \in \{1, \ldots, l\}, k_1 \text{ not necessarily distinct}) \), we have

a) \( B_iB_i^t = a_i b_i J_{mn} \)

b) \( B_iB_j + B_jB_i = 0 \quad i \neq j \).

Proof  By straightforward verification.

COROLLARY  If \( A \) is an orthogonal design of order \( n \) and type \((s_1, \ldots, s_t)\) then there exists an orthogonal design of order \( 2n \) and type \((s_1, s_2, s_3, \ldots, s_t)\), \( s_i = 1 \) or 2.

Proof  Let \( N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and \( N_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \), and let \( A_1, \ldots, A_t \) be the \( n \times n \) matrices given by the existence of \( A \). We now use these matrices in the Proposition above.

PROPOSITION 7  Let \( F_1, F_2, \ldots, F_r \) be weighing matrices of order \( n \) and weights \( s_1, s_2, \ldots, s_r \) \((s_k \geq 0, 1 \leq k \leq r)\) where

a) \( F_i \) is skew-symmetric and \( F_i \) is symmetric, for all \( i > 1 \).

b) \( F_i F_j = F_j F_i \) for all \( 1 \leq i, j \leq r \).

Let \( m \) be an integer for which \( \rho(m) \geq r \). Then there is an orthogonal design of order \( mn \) and type \((1, s_1, \ldots, s_r)\) on the \( 1 + r \) variables \( x_0, x_1, \ldots, x_r \).

Proof  Let \( D \) be an orthogonal design of order \( m \) and type \((1, 1, \ldots, 1)\) on the variables \( z_1, z_2, \ldots, z_{\rho(m)} \) as described in Section 2. Set \( z_1 = x_0 + x_1 F_1 \) and \( z_j = x_j F_j(1 \leq j \leq r) \) and set all the remaining variables (if any)
equal to the zero matrix. The resulting matrix, of order \(mn\), is the required
design since \(z_i^j z_j^i = z_i^i z_j^j\) for \(1 \leq i, j \leq \rho(m)\).

**Remarks** Note that \(F_2, \ldots, F_r\) may all be chosen to be the same matrix.
If \(r \geq 4\) the matrices \(x_0 I + x_1 F_1, x_2 F_2, x_3 F_3, x_4 F_4\) may be used in any
Baumert-Hall array; while if \(r \geq 8\), these matrices, along with \(x_5 F_5, x_6 F_6, x_7 F_7, x_8 F_8\) may be used in any Plotkin array.

**Definition** The \(l \times l\) matrices \(W\) and \(M\) are called amicable Hadamard
matrices if

i) \(W = I + S\) where \(S' = -S\) and \(M = M'\) are both Hadamard matrices
i.e. weighing matrices of weight \(l\), and

ii) \(WM' = MW'\).

These pairs of matrices exist for many orders (see [16; Part 4, Chapter II]
for a thorough discussion and also [15] for more recent results).

**Corollary 1** Let \(W\) and \(M\) (as above) be amicable Hadamard matrices
of order \(l\) and let \(n\) be any integer. Then there is an orthogonal design of order \(nl\)
of type \((l, l-1, l, 1, \ldots, l)\) on the variables \(x_0, x_1, x_2, \ldots, x_{\rho(n)}\).

**Corollary 2** If there is a Baumert-Hall array of order \(4n\) and amicable
Hadamard matrices of order \(l\) then there is an orthogonal design of order \(4nl\) and
of type \((n, nl-1, nl, nl, nl)\).

**Corollary 3** Let \(A_1, \ldots, A_s\) be an \(n \times n\) \((0, 1, -1)\) matrices as in
Proposition 6. Let \(W = I + S\) and \(M\) be amicable Hadamard matrices of order \(l\).
Then

\[
B_1 = A_1 \otimes I_1, \quad B_2 = A_1 \otimes S, \quad B_3 = A_2 \otimes M, \ldots, B_{s+1} = A_s \otimes M
\]

are matrices of order \(nl\) which give an orthogonal design of type \((a_1, a_1(l-1), a_2 l, \ldots, a_l)\) on the variables \(x_1, \ldots, x_n, x_{n+1}\).

**Corollary 4** Let \(W, M\) be amicable Hadamard matrices of order \(l\). If there
is a Plotkin array of order \(8n\) then there is an orthogonal design of order \(8nl\)
and type \((n, nl-1, nl, \ldots, nl)\) on the variables \(x_1, x_2, x_3, \ldots, x_9\).

The following results are well known but we do not know their first appearances.

**Proposition 8** Suppose there exist two circulant matrices \(A, B\) of order \(n\)
satisfying \(AA^T + BB^T = fI_n\). Then

\[
\begin{bmatrix}
A & B \\
-B^T & A^T
\end{bmatrix}
\]

is a \(W(2n, f)\)

when \(A, B\) are \((0, 1, -1)\) matrices and an orthogonal design of order \(2n\) and
type \((x_1, x_2, \ldots, x_n)\) on \(x_1, \ldots, x_n\) when \(f = \sum_{i=1}^n s_i x_i^2\).

**Proof** Straightforward verification.
PROPOSITION 9 Suppose there exist two circulant matrices \( A, B \) of order \( n \) satisfying
\[
AA' + BB' = fI_n.
\]
Further suppose that \( R \) is the back diagonal matrix. Then
\[
G = \begin{bmatrix}
A & BR \\
- BR & A
\end{bmatrix}
\]
is a \( W(2n, f) \) when \( A, B \) are \((0, 1, -1)\) matrices and an orthogonal design of order \( 2n \) and type \((s_1, s_2, \ldots, s_r)\) on \( x_1, \ldots, x_r \) when \( f = \sum_{i=1}^r s_i x_i^2 \).

Proof A straightforward verification.

Remark We note here that these propositions remain true if \( A \) and \( B \) are type 1 matrices and \( R \) is an appropriately chosen matrix. (See [16; Part 4, Chapter I] for a discussion of these matrices which generalize the notion of a circulant matrix.) A similar remark holds for Constructions 10–16.

DEFINITION We say that the weighing matrix \( W = W(n, k) \) is constructed from two circulant matrices \( M, N \) of order \( n \), if
\[
W = \begin{bmatrix}
M & N \\
- N' & M'
\end{bmatrix}.
\]

CONSTRUCTION 10 Suppose there is an \( W(n, k) \) constructed from two circulant matrices \( M, N \) of order \( n/2 \) with the property that \( M \ast N = 0 \) (\( \ast \) denotes Hadamard product). Then \( A = x_1 M + x_2 N, B = x_1 N - x_2 M \) may be used in Proposition 9 to obtain an orthogonal design of order \( n \) and type \((k, k)\) on \( x_1, x_2 \).

Proof Is a straightforward verification. One need only observe that since \( M, N \) are circulant \( M \ast R, N \ast R \) are back circulant; and if \( X \) is circulant and \( Y \) is back circulant then \( XY' = YX' \).

CONSTRUCTION 11 Suppose there exist \( W(n, k_i), i = 1, 2 \), constructed from circulant matrices \( M_i, N_i, i = 1, 2 \), of order \( n/2 \) where \( M_1 \ast M_2 = N_1 \ast N_2 = 0 \) and \( M_1 M_2' + M_2 M_1' = N_1 N_2' + N_2 N_1' = 0 \), then
\[
A = x_1 M_1 + x_2 M_2, \quad B = x_1 N_1 + x_2 N_2
\]
may be used in Proposition 9 to give an orthogonal design of order \( n \) and type \((k_1, k_2)\) on the variables \( x_1, x_2 \).

LEMMA 12 If
\[
N = \begin{bmatrix}
x_1 A_1 + x_2 A_2 & (x_1 A_3 + x_2 A_4)R \\
-(x_1 A_3 + x_2 A_4)R & (x_1 A_1 + x_2 A_2)
\end{bmatrix}
\]
is an orthogonal design of type \((s_1, s_2)\) where \( A_i, i = 1, 2, 3, 4 \), are \((0, 1, -1)\) matrices which are circulant and \( A_i J = y_i J \) (\( J \) is the matrix of ones), \( i = 1, \ldots, 4 \), then
\[
\sum_{i=1}^4 y_i^2 = s_1 + s_2.
\]
Proof. By definition,

\[(x_1A_1 + x_2A_2)(x_1A_1^t + x_2A_2^t) + (x_1A_3 + x_2A_4)(x_1A_3^t + x_2A_4^t) = (s_1x_1^2 + s_2x_2^2)I.\]

So,

\[x_1^2(A_1A_1^t + A_3A_3^t) + x_2^2(A_2A_2^t + A_4A_4^t) = (s_1x_1^2 + s_2x_2^2)I\]

and setting \(x_1 = x_2 = 1\) we have

\[\sum_{i=1}^{4} A_iA_i^t = (s_1 + s_2)I.\]

Post multiplying by \(J\) gives \((y_1^2 + \ldots + y_2^2)J = (s_1 + s_2)J\).

Remark. This lemma shows that for order \(N = 2n, n\) odd, this method can only construct orthogonal designs of order \(2n\) and type \((1, s)\) when \(s\) is a square. For under those circumstances we may assume \(A_1 = I, A_2 = -A_2\) is circulant, and \(A_3 = 0\). Hence \(y_1 = 1, y_2 = 0\) (since \(n\) is odd), and \(y_3 = 0\). Thus \(1 + y_2^2 = 1 + s\). So, for example, this method could not be used to construct an orthogonal design of type \((1, 8)\) in order \(2n, n\) odd. In fact we had shown that type \((1, 8)\) in order 10 is impossible by any method, although this was not precluded by our original considerations.

This suggests the following conjecture:

If \(n \equiv 2\,(\text{mod } 4)\) there does not exist a \((0, 1, -1)\) matrix \(X\) of order \(n\) such that \(X = -X^t\) and \(XX^t = kI_n\) unless (possibly) if \(k\) is a square. We verify this conjecture in Section 4.

Theorem 13 (Goethals and Seidel, see [16, p. 355]). Suppose there exist four circulant matrices \(A, B, C, D\) of order \(n\) satisfying

\[AA^t + BB^t + CC^t + DD^t = fI_n,\]

Let \(R\) be the back-diagonal matrix. Then

\[GS = \begin{bmatrix}
A & BR & CR & DR \\
-BR & A & D'R & -C'R \\
-CR & -D'R & A & B'R \\
-DR & C'R & -B'R & A
\end{bmatrix}\]

is a \(W(4n, f)\) when \(A, B, C, D\) are \((0, 1, -1)\) matrices, and an orthogonal design of order \(4n\) and type \((s_1, s_2, \ldots, s_t)\) on \(x_1, x_2, \ldots, x_t\) when

\[f = \sum_{j=1}^{t} s_jx_j^2.\]

Further \(GS\) is skew or skew-type if \(A\) is skew or skew-type.

We shall, of course, try to use Theorem 13 to construct orthogonal designs, but we first remark on the limitations of this method.

Lemma 14. Let \(A_i, i = 1, 2, 3, 4\) be circulant matrices of order \(n\) where

\[\sum_{i=1}^{4} A_iA_i^t = \left(\sum_{j=1}^{4} s_jx_j^2\right)I_n.\]
Suppose $A_i = \Sigma_{j=1}^l x_{ij}A_{ij}$ and that $A_iJ = y_{ij}J$. Then

$$\sum_{i=1}^4 \left( \sum_{j=1}^l y_{ij}^2 \right) = \sum_{j=1}^l s_j.$$ 

Proof. See Lemma 12 above.

Remark. If we have four circulants $A_1, A_2, A_3, A_4$ such that $\Sigma_{i=1}^4 A_iA_i^t = (x_1^2 + sx_2^2)I_n$, $n$ odd, then $s$ must be the sum of three squares. For we may assume $A_1 = x_1I + x_2A_{12}$, $A_2 = x_2A_{22}$, $A_3 = x_3A_{33}$, and $A_4 = x_4A_{44}$. Then, by Lemma 14 we have $1 + y_{12}^2 + y_{22}^2 + y_{32}^2 + y_{33}^2 = 1 + s$. But $A_{12} = -A_{12}$ and the order of $A_{12}$ is $n$ (odd). So $y_{12} = 0$, and consequently $s$ is a sum of three squares. We note that no $s$ of the form $4^t(8b + 7)$ can be written as the sum of three squares [6].

This remark led us to conjecture that if $n = 4t$, $t$ odd and $X = -X^t$ is a $W(n,k)$ then $k$ must be the sum of three squares. We verify this conjecture also in Section 4.

Construction 15. Suppose there exist $W(n,k_i)$, $i = 1, 2$ constructed from circulant matrices $M_i, N_i$, $i = 1, 2$ of order $n/2$. Then there exists an orthogonal design of order $2n$ and type $(k_1, k_2)$.

Proof. Set $A = x_1M_1$, $B = x_1N_1$, $C = x_2M_2$, $D = x_2N_2$ in Theorem 13.

Construction 16. Suppose there exist orthogonal designs $X_1, X_2$ of order $2n$ and type $(s_{11}, s_{12}, \ldots, s_{1m_1})$ on the variables $x_{11}, x_{12}, \ldots, x_{1m_1}$, $i = 1, 2$ each of which is constructed by using Proposition 9. Then there exists an orthogonal design of order $4n$ and type $(s_{11}, s_{12}, \ldots, s_{1m_1}, s_{21}, s_{22}, \ldots, s_{2m_2})$ on the variables $x_{11}, x_{12}, \ldots, x_{1m_1}, x_{21}, x_{22}, \ldots, x_{2m_2}$.

Proof. Let $A_i, B_i$ be the matrices used in Proposition 9 to form the orthogonal design $X_i$. Then use $A_1, B_1, A_2, B_2$ in the Goethals-Seidel array to get the result.

Definition. An L-family of order $n$ is a collection of $n \times n$ matrices $L_1, L_2, \ldots, L_4$ satisfying the following properties:

i) $L_i = -L_i$ for $1 \leq i \leq k - 1$

ii) $L_k = L_k$

iii) $L_iL_i = JL$ for $1 \leq j \leq k$

iv) $L_iL_k = L_kL_i$ for $1 \leq i \leq k - 1$.

Theorem 17. (a) If there is an L-family of order $n$ having $k$ members then there is an L-family of order $2n$ having $2k$ members.
b) If \( n = 2^t \) then there is an \( L \)-family of order \( n \) having \( n \) members.

**Proof** (b) follows from (a) and the observation that
\[
A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
\]
is an \( L \)-family of order 2.

To prove (a) we let \( L_1, \ldots, L_k \) be an \( L \)-family of order \( n \) and define \( L_1, L_2, \ldots, L_4, \ldots, L_{2k} \) to be matrices of order \( 2n \) where
\[
L_i = \begin{bmatrix} L_i & 0 \\ 0 & L_i \end{bmatrix} \quad \text{for } 1 \leq i \leq k - 1
\]
\[
L_k = \begin{bmatrix} 0 & L_k \\ -L_k & 0 \end{bmatrix}
\]
\[
L_{k+i} = \begin{bmatrix} L_i & L_k \\ -L_k & L_i \end{bmatrix} \quad \text{for } 1 \leq i \leq k - 1
\]
and
\[
L_{2k} = \begin{bmatrix} L_k & -L_k \\ L_k & L_k \end{bmatrix}
\]

It is an easy verification that \( L_1, \ldots, L_{2k} \) is an \( L \)-family of order \( 2n \).

**Construction 18** Let \( R, S \) be \((0, 1, -1)\) matrices of order \( n \) satisfying \( RR^t = r I_n \), \( SS^t = s I_n \), \( R^t = -R \), \( S^t = -S \) and \( RS^t = SR^t \). Then
\[
\begin{bmatrix} x_1 I + x_2 R & x_3 I + x_4 S \\ -x_3 I + x_4 S & x_1 I - x_2 R \end{bmatrix}
\]
is an orthogonal design of order \( 2n \) and type \((1, r, 1, s)\).

**Proof** Straightforward verification.

**Construction 19** Let \( P, Q, R, H \) be \((0, 1, -1)\) matrices of order \( n \) satisfying \( P^t = -P \), \( Q^t = -Q \), \( R^t = -R \), \( H^t = H \) and \( MN^t = NM^t \) for \( M, N \in \{P, Q, R, H\} \). If \( PP^t = p I_n \), \( QQ^t = q I_n \), \( RR^t = r I_n \) and \( HH^t = h I_n \) then
\[
\begin{bmatrix} x_1 I_n + x_2 P & x_3 I_n + x_4 Q & x_5 I_n + x_6 R & x_7 H \\ -x_3 I_n + x_4 Q & x_1 I_n - x_2 P & -x_7 H & x_5 I_n + x_6 R \\ -x_5 I_n + x_6 R & x_7 H & x_1 I_n - x_2 P & -x_3 I_n - x_4 Q \\ -x_7 H & -x_5 I_n + x_6 R & x_3 I_n - x_4 Q & x_1 I_n + x_2 P \end{bmatrix}
\]
is an orthogonal design of order \( 4n \) and type \((1, p, 1, q, 1, r, h)\).

**Remark** We note that if there are amicable Hadamard matrices \( A, B \) of order \( n \) where \( A = I_n + C \), \( C = -C^t \), then \( P = Q = R = C \) and \( H = B \) may be used in Construction 19. Also, if there is an \( L \)-family of order \( n \) then we may choose \( P = Q = R \) to be any member of the \( L \)-family (except the final member) and \( H \) equal to the final member and use them in Construction 19.
CONSTRUCTION 20  For \( n = 2 \) the following pairs of matrices may be used in Construction 19 (by choosing \( P = Q = R \) to be the first member of the pair and \( H \) to be the second).

\[
\begin{bmatrix}
0 & + \\
- & 0
\end{bmatrix}, \begin{bmatrix}
+ & 0 \\
0 & -
\end{bmatrix}
\quad \text{and} \quad \begin{bmatrix}
0 & + \\
- & 0
\end{bmatrix}, \begin{bmatrix}
+ & + \\
- & -
\end{bmatrix}.
\]

CONSTRUCTION 21  For \( n = 4 \), the following triplets of matrices may be used in Construction 19 (where \( P, Q, R \) are chosen from among the first two members of the triplet and \( H \) is chosen as the last member).

Let \( A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), \( A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \), \( A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), \( I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and

\( A_4 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \), then

I. \( \{ A_1 \otimes A_2, (A_2 \otimes I_2) + (A_4 \otimes A_1), A_4 \otimes A_2 \} \).

II. \( \{ A_1 \otimes A_2, A_4 \otimes A_1, A_4 \otimes A_2 \} \).

III. \( \{ A_1 \otimes A_2, (A_1 \otimes I_2) + (A_4 \otimes A_1), (A_3 \otimes A_2) + (I_2 \otimes A_3) + (A_4 \otimes A_1) \} \).

IV. \( \{ (I_2 \otimes A_2) + (A_1 \otimes A_3), (A_1 \otimes I_2) + (A_4 \otimes A_3), (A_3 \otimes A_2) + (I_2 \otimes A_3) + (A_4 \otimes A_1) \} \).

V. \( \{ A_1 \otimes A_2, (A_1 \otimes I_2) + (A_4 \otimes A_1), (A_2 \otimes I_3) + (A_3 \otimes A_2), (I_2 \otimes A_3) + (A_4 \otimes A_1) \} \).

VI. \( \{ (I_2 \otimes A_1) + (A_1 \otimes A_3), (A_1 \otimes I_2) + (A_4 \otimes A_1), (A_3 \otimes A_2) + (I_2 \otimes A_3) + (A_4 \otimes A_1) \} \).

(Where \( \otimes \) denotes direct sum and \( \odot \) denotes the Kronecker product).

CONSTRUCTION 22  The following pairs of orthogonal designs satisfy the equation \( XY' = YX' \),

I. \( X = \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix} \), \( Y = \begin{bmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{bmatrix} \)

II. \( X = \begin{bmatrix} x_1 & x_2 & x_3 & x_3 \\ -x_2 & x_1 & x_3 & -x_3 \\ x_3 & x_3 & -x_1 & -x_2 \\ x_3 & -x_3 & x_2 & -x_4 \end{bmatrix} \), \( Y = \begin{bmatrix} y_1 & y_2 & y_3 & y_3 \\ y_2 & -y_1 & y_3 & -y_3 \\ -y_3 & -y_3 & y_2 & y_1 \\ -y_3 & y_3 & y_1 & -y_2 \end{bmatrix} \).
COROLLARY  If there is an orthogonal design of order \( n \) and type \( (s_1, s_2) \) then
i) there is an orthogonal design of order \( 2n \) and type \( (s_1, s_1, s_2, s_2) \)
ii) there is an orthogonal design of order \( 4n \) and type \( (s_1, s_1, 2s_1, s_2, s_2, 2s_2) \).

4. SOME MORE NECESSARY CONDITIONS

PROPOSITION 23  If \( n \) is odd then a necessary condition that \( W(n, k) \) exist is that
\[
(n - k)^2 - (n - k) + 2 > n.
\]

Proof  We have already noted that \( k \) must be a square. We consider two
cases:

Case 1  \( k \) even.

We may as well assume the first row of \( A \) is
\[
[\begin{array}{cccc}
1 & \cdots & 1 & 0 \cdots 0 \\
\hline
k & & & \\
& & & \\
& & & \\
\end{array}
\]_{n-k}
\]

Since \( n \) is odd and \( k \) is even, \( n - k \) is odd. Since every row contains an even
number of nonzero entries and an even number of nonzero entries must
appear under the 1's of the first row we find that there must be an odd number
of zeroes (and hence at least one) under the zeroes of the first row. Thus, if
there are \( (n - k)(n - k - 1) + 2 \) rows then at least one of the last \( n - k \)
columns contains more than \( n - k \) zeroes. Thus, in order that \( W(n, k) \) exist
we must have \( (n - k)(n - k - 1) + 2 > n \).

Case 2  \( k \) odd.

As in Case 1, we may assume the first row is
\[
[\begin{array}{cccc}
1 & \cdots & 1 & 0 \cdots 0 \\
\hline
k & & & \\
& & & \\
& & & \\
\end{array}
\]_{n-k}
\]

Since \( k \) is odd, \( n - k \) is even. But, any other row must have an even number
of nonzero elements under the \( k \) ones of the first row. This leaves an odd
number of nonzero elements to go in the last \( n - k \) columns. Thus, there
must be at least one zero in every row among the last \( n - k \) columns. The
argument now proceeds as in Case 1. So, we conclude that \( W(n, k) \) does not
exist if \( n \geq (n - k)^2 - (n - k) + 2 \).

Remark  We conclude from this theorem that \( W(5, 4), W(11, 9), W(19, 16),
W(29, 25), W(36, 41) \) etc. do not exist. This does not cover \( W(13, 9) \) or
\( W(31, 25), \ldots, W(111, 100) \) among others.

We are indebted to Professor David Gregory for the construction of
\( W(13, 9) \) which we exhibit below.
ORTHOGONAL DESIGNS

\[
\begin{bmatrix}
0 & 0 & 0 & - & - & + & + & + & + \\
0 & + & + & + & 0 & 0 & - & - & + & + \\
0 & + & + & - & - & 0 & 0 & 0 & - & - \\
0 & - & - & - & - & - & - & 0 & 0 & 0 \\
+ & 0 & + & + & + & - & - & + & 0 & + \\
+ & 0 & + & - & 0 & + & + & 0 & + & 0 \\
+ & 0 & + & 0 & - & 0 & + & 0 & + & 0 \\
+ & 0 & - & 0 & + & 0 & - & 0 & + & 0 \\
+ & 0 & - & - & 0 & + & 0 & 0 & + & 0 \\
+ & - & - & + & 0 & 0 & 0 & - & - & 0 \\
+ & - & 0 & 0 & 0 & - & - & 0 & 0 & 0 \\
+ & + & - & - & 0 & 0 & 0 & - & - & 0 \\
+ & + & - & 0 & 0 & + & + & 0 & 0 & 0 \\
+ & + & 0 & 0 & - & 0 & 0 & - & 0 & 0 \\
+ & - & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 \\
+ & - & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \\
+ & + & - & - & 0 & 0 & 0 & 0 & 0 & 0 \\
+ & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

We have also been able to construct a circulant \(W(13, 9)\). It’s first row is

\[
[0 \ 0 \ 1 \ 0 \ -1 \ 1 \ 1 \ 1 \ 1 \ 0 \ -1 \ 1 \ -1].
\]

The “boundary” values from this Proposition are of special interest, i.e. if \(n = (n - k)^2 - (n - k) + 1\). An inspection of the proof shows that if a \(W(n, k)\) exists for such an \(n\) and \(k\) then the incidence of zeroes between any two rows is exactly one. So, if we let \(A = W(n, k)\) and let \(B = J - A \ast A\) then \(B\) satisfies \(BB^t = (n - k - 1)J_n + 1 \cdot J_n\), i.e. \(B\) is the incidence matrix of a projective plane of order \(n - k - 1\). So, the existence of \(W(111, 100)\) would imply the existence of a projective plane of order 10. Thus, the Bruck-Ryser Theorem [5 or 10] on the nonexistence of projective planes of various orders imply nonexistence for the appropriate \(W(n, k)\). However, even when the projective plane of order \((n - k - 1)\) exists the construction of \(W(n, k)\) remains a formidable task.

We have previously mentioned two conjectures on the nonexistence of skew-symmetric \(W(n, k)\) for various \(k\) and \(n\). We repeat these conjectures here for definiteness.

**Conjecture 1** If \(n \equiv 2 (\bmod \ 4)\) and \(X\) is a \(W(n, k)\) for which \(X = -X^t\), then \(k\) is a square.

**Conjecture 2** If \(n = 4t, t \text{ odd and } X\) is a \(W(n, k)\) for which \(X = -X^t\), then \(k\) must be a sum of three squares.

As far as conjecture 1 is concerned, we prove more generally

**Proposition 24** If \(n \equiv 2 (\bmod \ 4)\) and \(X\) is a skew-symmetric rational matrix of order \(n\) satisfying \(XX^t = kI_n\), then \(k = q^2\) for some rational number \(q\). If, in addition, \(k\) is an integer then \(k = a^2\) for some integer “\(a\)”.

**Proof** Since \(XX^t = kI_n\) we have \(\det X = (k^2)^{n/2} = k^{n^2}\) where \(n/2 = s\) is odd. On the other hand, since \(X\) is skew-symmetric, \(\det X = r^2\) where \(r = \text{Pfaffian of } X\) (see [1, pages 141–142]). Thus, \(r^2 = k^s\), and since \(s\) is odd,
$k$ must be a square. To complete the proof we just note that if an integer is the square of a rational number it is also the square of an integer.

**Corollary** If $n \equiv 2(\text{mod } 4)$ and there is an orthogonal design of type $(a, b)$ in order $n$ then $b/a$ is a square.

**Proof** We let $X$ be the orthogonal design and write $X = x_1A_1 + x_2A_2$. Then we have shown that $A_1A_1^* = aI_n$, $A_2A_2^* = bI_n$ and $A_1A_2^* + A_2A_1^* = 0$.

Consider the matrices $B_1 = (1/a)A_1^*A_1$, $B_2 = (1/a)A_2^*A_2$. Then $B_1 = I_n$ and $B_1B_2 + B_2B_1 = 0$ and so $B_2$ is skew-symmetric. Furthermore, $B_2B_2^* = (b/a)I_n$. Thus by Proposition 24, $b/a$ must be a square.

**Example** There is no orthogonal design of type $(4, 5)$ or $(1, 8)$ in order 10 and none of type $(5, 8)$ in order 14. The first two are clear; to see the result for $(5, 8)$ we note that $\frac{5}{8} = \frac{5}{8}$ and 40 is not a square of an integer. Note that the existence of these designs was not prohibited by the corollary to Proposition 3.

The proof of Conjecture 2 closely parallels the interesting proof of a theorem of Raghavaroa (our Proposition 3) which was given by van Lint and Seidel in [13]. We shall make extensive use of the linearity properties of the Pfaffian and we refer the reader to Artin’s monograph [1, p. 141] for the salient facts about this matrix function. Our original proof has been vastly improved by the following lemma, which was supplied by N. J. Pullman. We are very grateful to him for it.

**Lemma** If $M$ and $A$ are skew-symmetric matrices of order $n$ and $\mathcal{O}$ is the set of all diagonal matrices of order $n$ whose diagonal entries are all in $\{1, -1\}$ then

\[
\sum_{D \in \mathcal{O}} P f(DMD + A) = 2^{n-1}(P f(A) + P f(M))
\]

and

\[
\sum_{D \in \mathcal{O}} P f(DMD + A) = 2^{n-1}(P f(A) - P f(M)).
\]

**Proof** Part (b) is equivalent to (a) for if we use (a) with $D_0MD_0$ instead of $M$ and $\det D_0 = -1$ then (b) follows directly from (a); similarly (b) implies (a). We shall prove (a) by induction on $n$ which we may assume to be even. If $n = 2$ then

\[
\sum_{D \in \mathcal{O}} P f(DMD + A) = P f(1M1 + A) + P f((-1)M(-1) + A)
\]

\[
= 2P f(M + A) = 2(m_{11} + a_{11})
\]

\[
= 2(P f(M) + P f(A)).
\]

Suppose (a) is true for $n - 2 \geq 0$. Let $X_{ij}$ denote the matrix obtained from
\( X \) be deleting rows \( i \) and \( j \) and columns \( i \) and \( j \). Note that \([DXD]_{ij} = D_{ij}X_{ij}D_{ij}\) when \( D \) is a diagonal matrix \( d_1, d_2, \ldots, d_n\).

\[
\sum_{D \in \mathcal{D}} Pf(DMD + A)
\]

\[
= \sum_{D \in \mathcal{D}} \sum_{j=2}^{n} (-1)^j (d_1m_{1j}d_j + a_{1j})Pf(DMD + A)_{ij}
\]

\[
= \sum_{j=2}^{n} (-1)^j \left\{ \sum_{D_{ij} \in \mathcal{D}} \left( m_{ij} + a_{ij} \right)Pf(DMD + A)_{ij} + \sum_{D_{ij} \in \mathcal{D}} \left( a_{ij} - m_{ij} \right)Pf(DMD + A)_{ij} \right\}
\]

\[
= \sum_{j=2}^{n} 2(-1)^j \left\{ \sum_{D_{ij} \in \mathcal{D}} \left( m_{ij} + a_{ij} \right)Pf(D_{ij}M_{ij}D_{ij} + A_{ij}) + \sum_{D_{ij} \in \mathcal{D}} \left( a_{ij} - m_{ij} \right)Pf(D_{ij}M_{ij}D_{ij} + A_{ij}) \right\}
\]

\[
= \sum_{j=2}^{n} 2(-1)^j \left\{ \sum_{D_{ij} \in \mathcal{D}} \left( m_{ij} + a_{ij} \right)Pf(D_{ij}M_{ij}D_{ij} + A_{ij}) + \sum_{D_{ij} \in \mathcal{D}} \left( a_{ij} - m_{ij} \right)Pf(D_{ij}M_{ij}D_{ij} + A_{ij}) \right\}
\]

\[
= \sum_{j=2}^{n} 2(-1)^j \left\{ \sum_{D_{ij} \in \mathcal{D}} \left( m_{ij} + a_{ij} \right)2^{-3}(Pf(A_{ij}) + Pf(M_{ij})) + \sum_{D_{ij} \in \mathcal{D}} \left( a_{ij} - m_{ij} \right)2^{-3}(Pf(A_{ij}) - Pf(M_{ij})) \right\}
\]

\[
= 2^{n-1} \sum_{j=2}^{n} \left( (-1)^j (a_{ij}Pf(A_{ij}) + m_{ij}Pf(M_{ij})) \right)
\]

\[
= 2^{n-1}(Pf(A) + Pf(M)).
\]

**Corollary 1**

a) \( Pf(A) = 2^{-n} \sum_{D \in \mathcal{D}} Pf(DMD + A) \)

b) \( Pf(M) = 2^{-n} \sum_{D \in \mathcal{D}} (\det D)Pf(DMD + A) \).

**Corollary 2** If \( A \) or \( M \) is nonsingular then there exists a \( D \in \mathcal{D} \) for which \( DMD + A \) is nonsingular.

**Proposition 25** Let \( X \) be a rational skew-symmetric matrix of order \( n = 4s \), \( s \) odd and suppose \( XX' = kI_s \). Then \( k = q_1^2 + q_2^2 + q_3^2 \), \( q_i \in \mathcal{Q} \). If, in addition \( k \in \mathbb{Z} \) then \( q_1, q_2, q_3 \) may also be chosen in \( \mathbb{Z} \).
Proof Our proof is by induction on s. If s = 1, then
\[
X = \begin{bmatrix}
0 & x_1 & x_2 & x_3 \\
-x_1 & 0 & x_4 & x_5 \\
-x_2 & -x_4 & 0 & x_6 \\
-x_3 & -x_5 & -x_6 & 0
\end{bmatrix}
\]
and if \(XX' = kI_4\), then \(k = x_1^2 + x_2^2 + x_3^2, x_i \in \mathbb{Q}\), proving the first assertion of the Proposition. Now, if \(k \in \mathbb{Z}\) then by the theorem of Cassels-Davenport [see 11, p. 46] we obtain \(k = a^2 + b^2 + c^2\) for three integers \(a, b, c\).

So, we now assume \(s \text{ odd, } s > 1\). By the theorem of Lagrange we may certainly write \(k = k_1^2 + k_2^2 + k_3^2 + k_4^2\), where \(k_i \in \mathbb{Q}\).

Let
\[
\tilde{M} = \begin{bmatrix}
k_1 & -k_2 & -k_3 & -k_4 \\
k_2 & k_1 & -k_4 & k_3 \\
k_3 & -k_4 & k_1 & -k_2 \\
k_4 & k_3 & -k_2 & k_1
\end{bmatrix}
\]
and let
\[
M = \begin{bmatrix}
0 & \tilde{M} \\
-\tilde{M}^t & 0
\end{bmatrix}.
\]
Then, \(M\) has order 8, \(M = -M^t \text{ and } MM^t = kI_8\). Furthermore, if \(D\) is any diagonal matrix of order 8, with diagonal entries \(\pm 1\), then \(M = DMD\) is still skew-symmetric, has order 8 and satisfies \(MM^t = kI_8\).

We partition the matrix \(X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\) where \(A\) has order 8 and note that \(A\) is skew-symmetric. Thus, since \(M\) and \(A\) are skew-symmetric and \(M\) is nonsingular there is, by Corollary 2 above, a diagonal matrix \(D \in \mathcal{D}\) such that \(DMD + A\) is nonsingular. (For convenience we shall still refer to \(DMD\) as \(M\)).

Now, let \(P = D - C(M + A)^{-1}B\). Clearly \(P\) is a matrix of order \(n - 8\) and one checks easily that \(P\) is skew-symmetric. We further claim that \(PP^t = kI_{n-8}\). This is easily verified by calculating the following matrix product in two ways:

If \(N = M + A\), consider
\[
\begin{bmatrix} -B(N^{-1})^{-1} & I_{n-8} \end{bmatrix} \begin{bmatrix} A & B^t \\ C^t & D^t \end{bmatrix} \begin{bmatrix} A & B^t \\ C & D \end{bmatrix} \begin{bmatrix} -N^{-1}B^t \\ I_{n-8} \end{bmatrix}
\]
first by grouping the product as \((ST)(UV)\) and then as \((STU)V\).

Thus, the induction hypothesis may be applied to \(P\) to complete the proof.

Corollary If there is an orthogonal design of type \((a, b)\) in order \(n = 4s, s \text{ odd,}\) then \(b/a\) is a sum of \(\leq 3\) squares of rational numbers.

Proof Exactly the same as the proof of the Corollary to Proposition 24.
Example There is no orthogonal design of type (1, 7), (4, 7) or (3, 5) in order 12. It is well known that 7 is not the sum of three integer squares. Now $\frac{7}{2} = \frac{15}{3}$ and 15 is not the sum of three integer squares and so $\frac{7}{2}$ is not the sum of three rational squares.

5. SOME CALCULATIONS

If $n \equiv 1 \text{ or } 3 \pmod{4}$

$n = 5$ By Proposition 23 no weighing matrix exists (except for weight 1 which we will consistently ignore).

$n = 7$ The only possibility is a $W(7, 4)$. The row $[- + + 0 + 0 0]$ generates a circulant $W(7, 4)$, which we will call $A$. $J - A \cdot A$ is the incidence matrix of the projective plane of order 2.

$n = 9$ The only possibility is a $W(9, 4)$. A tedious, but not difficult, check on what the upper left hand corner of such a matrix must be, shows that $W(9, 4)$ does not exist.

$n = 11$ Weight 9 is precluded by Proposition 23, so we need only discover a $W(11, 4)$. If $A$ is a $W(7, 4)$ and $H_4$ an Hadamard matrix of order 4 then $A \oplus H_4$ is a $W(11, 4)$.

$n = 13$ The possible weights are 4 and 9. We have already exhibited a $W(13, 9)$. In [14] a $W(6, 4)$ was found, so $W(6, 4) \oplus W(7, 4) = W(13, 4)$.

We mention, that $W(n, 4)$ exists for all $n \geq 4$ except for $n = 5, 9$. The proof of this follows readily from the fact that if two integers are relatively prime then eventually every integer can be expressed as a positive linear combination of these two integers. Thus, from the existence of $W(4, 4)$, $W(6, 4)$ and $W(7, 4)$ we assert that the “eventually” we mentioned occurs

when $n = 10$. Hence by taking the appropriate direct sums of $W(4, 4)$, $W(6, 4)$ and $W(7, 4)$ we get $W(n, 4)$ for all $n \geq 10$. In fact, if $k = 4^s$ then there exists $N$ such that for all $n \geq N$ $W(n, k)$ exists. We suspect this may be true for any square weight. We have exhibited a $W(13, 9)$ and in [14] or [4] a $W(10, 9)$, $W(12, 9)$ and $W(16, 9)$ are constructed and so $W(n, 9)$ exists for $n \geq 32$. So, our remarks above apply equally well to powers of 9. It would be interesting to have least upper bounds in these cases also, as we had for 4.

$n \equiv 2 \pmod{4}$

In this case $\rho(n) = 2$ and so we will consider the existence of designs on $\leq 2$ variables.
All designs exist.

By Proposition 3 an orthogonal design of type \((s)\) or of type \((s_1, s_2)\) can exist only if \(s_1, s_2, s_1 + s_2\) are each a sum of less than or equal to two squares. The numbers less than or equal to 6 that are the sum of \(\leq 2\) squares are 1, 2, 4, 5.

By the corollary to Proposition 3 the only designs on two variables that might exist are types \((1, 1)\), \((1, 4)\) and \((2, 2)\). Using Proposition 9, we have

\[
\begin{array}{ccc|ccc}
\text{(type)} & \text{1st row of } A & \text{1st row of } B \\
(1, 1) & x_1 & 0 & 0 & x_2 & 0 & 0 \\
(2, 2) & x_1 & x_2 & 0 & x_2 & -x_1 & 0 \\
(1, 4) & x_1 & x_2 & -x_2 & 0 & x_2 & x_2 \\
\end{array}
\]

Now, by setting the variables in these designs equal to one or zero, we obtain \(W(6, k)\) for \(k = 1, 2, 4, 5\).

Thus, for \(n = 6\), Proposition 3 gives necessary and sufficient conditions for existence.

Proposition 3 eliminates many 2-tuples \((s_1, s_2)\) \((s_1 \leq s_2, s_1 + s_2 \leq 10)\) as the type of an orthogonal design of order 10. The 2-tuples \((1, 8)\) and \((4, 5)\) are eliminated by the corollary to Proposition 24. The remaining are the types of an orthogonal design and can be constructed using Proposition 9, viz.

\[
\begin{array}{ccc|ccc}
\text{(type)} & \text{1st row of } A & \text{1st row of } B \\
(1, 1) & x_1 & 0 & 0 & 0 & 0 & 0 \\
(1, 4) & 0 & 0 & x_2 & x_2 & 0 & x_1 \\
(2, 2) & x_1 & x_2 & 0 & 0 & x_2 & -x_1 & 0 & 0 \\
(4, 4) & 0 & x_1 & x_2 & -x_2 & x_1 & 0 \\
\end{array}
\]

The numbers \(\leq 10\) that are the sum of \(\leq 2\) squares are 1, 2, 4, 5, 8, 9. The designs above give \(W(10, k)\) for \(k = 1, 2, 4, 5, 8\). We obtain a \(W(10, 9)\) by using Proposition 9 and

\[
\begin{array}{ccc|ccc}
\text{(type)} & \text{1st row of } A & \text{1st row of } B \\
9 & 1 & -1 & -1 & -1 & -1 & 0 & 1 & -1 & -1 & 1 \\
\end{array}
\]

Thus, Proposition 3 and the corollary to Proposition 24 give necessary and sufficient conditions for the existence of orthogonal designs of order 10.

Again, Proposition 3 and the corollary to Proposition 24 exclude many 2-tuples \((s_1, s_2)\) \((where 0 < s_1 \leq s_2, s_1 + s_2 \leq 14)\) as the types of
orthogonal designs of order 14. The tuple (4, 9) is not excluded and we cannot decide if it is the type of an orthogonal designs of order 14. All the other 2-tuples are the types of orthogonal designs. We use Proposition 9, and

<table>
<thead>
<tr>
<th>(type)</th>
<th>1st row of $A$</th>
<th>1st row of $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>$x_1$ 0 0 0 0 0 0</td>
<td>$x_2$ 0 0 0 0 0 0</td>
</tr>
<tr>
<td>(1,4)</td>
<td>$x_1$ 0 0 $x_2$ $-x_2$ 0 0</td>
<td>0 0 0 0 0 0</td>
</tr>
<tr>
<td>(1,9)</td>
<td>$x_1$ $x_2$ $x_3$ $-x_2$ $x_2$ $-x_2$ $x_2$</td>
<td>0 $x_2$ $x_2$ 0 $x_3$ 0 0</td>
</tr>
<tr>
<td>(2,2)</td>
<td>$x_1$ $x_2$ 0 0 0 0 0</td>
<td>$x_3$ $-x_2$ 0 0 0 0 0</td>
</tr>
<tr>
<td>(2,8)</td>
<td>$-x_2$ $x_2$ $x_3$ 0 $x_3$ 0 $x_3$</td>
<td>$-x_3$ $x_2$ $x_3$ 0 $x_3$ 0 $-x_3$</td>
</tr>
<tr>
<td>(4,4)</td>
<td>$x_1$ $x_1$ $x_3$ $-x_2$ 0 0 0</td>
<td>$x_3$ $-x_1$ $x_2$ $x_2$ 0 0 0</td>
</tr>
<tr>
<td>(5,5)</td>
<td>$-x_2$ $x_2$ $x_3$ $x_3$</td>
<td>$-x_1$ $x_1$ $x_3$ $-x_2$ 0 0 0</td>
</tr>
</tbody>
</table>

The numbers \(\leq 14\) that are the sum of \(\leq 2\) squares are 1, 2, 4, 5, 8, 9, 10, 13. The designs above give $W(14, k)$ for $k = 1, 2, 4, 5, 8, 9, 10$. A $W(14, 13)$ is given in [14], thus the necessary condition given in Proposition 3, for the existence of a weighing matrix, is also sufficient for $n = 14$.

$n \equiv 0 \pmod{4}$

We shall first consider $n = 2^t$, $t \geq 2$.

$n = 4, 8$ is covered by Corollary 1 to Lemma 4.

$n = 16$ Now $\rho(16) = 9$ and we must consider designs on $\leq 9$ letters.

In [4] it was shown that if $n = 2^t$ then there exist $W(n, k)$ for all $k \leq n$. Thus we shall not make any further remarks about designs on one variable. From Theorem 17 we conclude that designs of type $(1, s)$ exist when $n = 2^t$ and $s < n$.

To obtain the other designs on two variables and of order 16 we use Proposition 6 and the corollary to it. We note that there is an orthogonal design of type $(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ and order 8. This gives orthogonal designs of order 16 and type $(e_1, e_2, \ldots, e_9)$ where $e_i = 1, 2$. Setting the variables in these designs equal to each other, or to zero we obtain (as the reader can easily check) 61 of the 64 possible designs on two variables in order 16. The remaining designs are of type $(3, 13), (5, 11)$ and $(7, 9)$.

Now, using Corollary 1 to Proposition 7 and the fact that there are amicable Hadamard matrices of order 4 we obtain an orthogonal design of order 16 and type $(1, 3, 4, 4, 4)$. Thus by setting the variables in this design equal to each other, as necessary, we recover designs of type $(3, 13), (5, 11)$ and $(7, 9)$. Thus all orthogonal designs of order 16 on two variables exist.

We now consider designs on three variables. By using Proposition 6, as we did above, we obtain all but 23 of the 123 possible designs on 3 variables. The missing 23 are the following types
Using Corollary 1 to Proposition 7 we obtain an orthogonal design of order 16 and type \((1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)\), since there are amicable Hadamard matrices of order 2. By setting some of the variables in this design equal to each other or to zero we obtain all those designs on three letters marked by \(\ast\).

Now use Construction 19 and the fact that there are amicable Hadamard matrices of order 4 to obtain an orthogonal design of type \((1, 1, 1, 3, 3, 3, 3)\). If we use Construction 21 (VI) we obtain an orthogonal design of order 16 and type \((1, 1, 1, 2, 3, 3, 3, 4)\). Thus all 3-tuples \((s_1, s_2, s_3)\) \((1 \leq s_1 \leq s_2 \leq s_3, s_1 + s_2 + s_3 \leq 16)\) are the types of orthogonal designs of order 16.

We next proceed to designs on four variables. Again, the use of Proposition 6 gives many of these designs, i.e. of the 155 possible 4-tuples \((s_1, s_2, s_3, s_4)\) \((\Sigma^4_{i=1} s_i \leq 16, 1 \leq s_1 \leq s_2 \leq s_3 \leq s_4)\) all but 51 are obtained as orthogonal designs by Proposition 6. The orthogonal designs \((1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)\), \((1, 1, 1, 3, 3, 3, 3, 4)\) and \((1, 1, 1, 2, 3, 3, 4)\) that we have already obtained give many of these 51. The designs not yet obtained are

\[(1, 1, 1, 11)\quad \text{and} \quad (1, 5, 5, 5)\]

If we let

\[
P = Q = R = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & - \\ - & 0 & 0 & + \\ - & + & - & 0 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} + & + & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & - & - \\ 0 & 0 & - & + \end{bmatrix}
\]

in Construction 19 we obtain an orthogonal design of type \((1, 1, 1, 2, 3, 3, 3)\) and order 16. This gives an orthogonal design of type \((1, 1, 1, 11)\) and order 16. We have been unable to construct an orthogonal design of order 16 and type \((1, 5, 5, 5)\).

We now consider designs on five variables. A tabulation shows that there are 149 possible 5-tuples, \((s_1, s_2, s_3, s_4, s_5)\) where \(\Sigma^5_{i=1} s_i \leq 16\) and \(1 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq s_5\). This time Proposition 6 gives only 68 of these tuples as the type of an orthogonal design. However, the designs \((1, 1, 1, 2, 2, 2, 2, 2, 2, 2)\), \((1, 1, 1, 3, 3, 3, 3, 4)\), \((1, 1, 1, 2, 3, 3, 3)\) and \((1, 1, 1, 2, 3, 3, 4)\) give all but the following 16:

\[
(1, 1, 1, 13) \quad (1, 3, 12) \ast \quad (1, 6, 9) \ast \\
(1, 1, 14) \ast \quad (1, 4, 11) \ast \quad (1, 7, 7) \\
(1, 2, 13) \ast \quad (1, 5, 9) \quad (1, 7, 8) \ast \\
(1, 3, 11) \quad (1, 5, 10) \ast \quad (2, 3, 11) \ast \\
(2, 5, 9) \ast \quad (3, 4, 9) \ast \quad (4, 5, 7) \ast \\
(2, 7, 7) \ast \quad (3, 5, 7) \quad (5, 5, 5) \\
(3, 3, 9) \quad (3, 5, 8) \ast \quad (5, 5, 6) \ast \\
(3, 3, 10) \ast \quad (3, 6, 7) \ast
\]
Use Construction 21 (V) to obtain \((1, 1, 1, 1, 3, 3, 3, 3)\) and \((1, 1, 1, 1, 3, 3, 3, 3)\); Construction 21 (III) to obtain \((1, 1, 1, 1, 3, 3, 3, 3)\) and \((1, 1, 1, 1, 3, 3, 3, 3)\) and Construction 21 (IV) to obtain \((1, 1, 1, 1, 2, 2, 3, 3, 3)\).

If we let
\[
P = Q = R = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & - \\ - & - & 0 & + \\ - & + & - & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & + & + & + \\ + & 0 & - & + \\ + & - & + & 0 \\ + & + & 0 & 0 \end{bmatrix}
\]
we obtain (from Construction 19) type \((1, 1, 1, 3, 3, 3, 3)\). Also, using Construction 21 (VI) we obtain \((1, 1, 1, 2, 2, 3, 4)\).

Now use Construction 22 (II) and the design \((1, 3)\) in order 4 to obtain \((1, 1, 2, 3, 3, 6)\).

The designs that can be obtained from the constructions above have been marked with a \((\vee)\). We have been unable to decide if the 9 remaining tuples are the types of orthogonal designs.

Our results get markedly worse here. Of the 125 6-tuples that might be the types of orthogonal designs we can construct only 88. For the reader’s information we list the 37 6-tuples for which the situation is undecided.

\[
\begin{array}{llll}
1, 1, 1, 1, 1, 8 & 1, 1, 1, 1, 5, 5 & 1, 1, 2, 2, 2, 7 \\
1, 1, 1, 1, 1, 9 & 1, 1, 1, 1, 5, 6 & 1, 1, 2, 2, 3, 6 \\
1, 1, 1, 1, 1, 10 & 1, 1, 1, 1, 5, 7 & 1, 1, 2, 2, 3, 7 \\
1, 1, 1, 1, 1, 11 & 1, 1, 1, 1, 6, 6 & 1, 1, 2, 2, 4, 5 \\
1, 1, 1, 1, 1, 12 & 1, 1, 1, 2, 2, 8 & 1, 1, 2, 2, 5, 5 \\
1, 1, 1, 1, 1, 8 & 1, 1, 1, 2, 2, 9 & 1, 1, 2, 3, 4, 5 \\
1, 1, 1, 1, 2, 9 & 1, 1, 1, 2, 3, 8 & 1, 2, 2, 2, 3, 5 \\
1, 1, 1, 1, 1, 10 & 1, 1, 1, 2, 4, 7 & 1, 2, 2, 3, 4, 5 \\
1, 1, 1, 1, 3, 8 & 1, 1, 1, 2, 5, 5 & 1, 3, 3, 3, 3, 3 \\
1, 1, 1, 1, 3, 9 & 1, 1, 1, 2, 5, 6 & 2, 2, 2, 3, 3, 3 \\
1, 1, 1, 1, 4, 5 & 1, 1, 1, 3, 5, 5 & 2, 2, 3, 3, 3, 3 \\
1, 1, 1, 1, 4, 7 & 1, 1, 1, 4, 4, 4 & \\
1, 1, 1, 1, 4, 8 & 1, 1, 1, 4, 4, 5 & \\
\end{array}
\]

Of the 94 possible 7-tuples we can only prove that 37 are the types of orthogonal designs. (We have indicated these constructions above.) Again, for the reader’s benefit we list the undecided tuples.
Of the 67 8-tuples that might be the types of orthogonal designs we can only prove that 11 are. The orthogonal designs on eight variables that we know are

\[
\begin{align*}
1, 1, 1, 1, 1, 1, 1, 1 & \quad 1, 1, 1, 1, 1, 3, 5 & 1, 1, 1, 2, 2, 2, 6 \\
1, 1, 1, 1, 1, 1, 1, 6 & \quad 1, 1, 1, 1, 1, 3, 6 & 1, 1, 1, 1, 2, 2, 7 \\
1, 1, 1, 1, 1, 1, 1, 7 & \quad 1, 1, 1, 1, 1, 3, 7 & 1, 1, 1, 1, 2, 2, 8 \\
1, 1, 1, 1, 1, 1, 1, 8 & \quad 1, 1, 1, 1, 1, 3, 8 & 1, 1, 1, 1, 2, 3, 3 \\
1, 1, 1, 1, 1, 1, 1, 9 & \quad 1, 1, 1, 1, 1, 4, 4 & 1, 1, 1, 1, 2, 3, 4 \\
1, 1, 1, 1, 1, 1, 1, 10 & \quad 1, 1, 1, 1, 1, 4, 5 & 1, 1, 1, 1, 2, 3, 5 \\
1, 1, 1, 1, 1, 1, 2, 5 & \quad 1, 1, 1, 1, 1, 4, 6 & 1, 1, 1, 1, 2, 3, 6 \\
1, 1, 1, 1, 1, 1, 2, 6 & \quad 1, 1, 1, 1, 1, 4, 7 & 1, 1, 1, 1, 2, 3, 7 \\
1, 1, 1, 1, 1, 1, 2, 7 & \quad 1, 1, 1, 1, 1, 5, 5 & 1, 1, 1, 1, 2, 4, 4 \\
1, 1, 1, 1, 1, 1, 2, 8 & \quad 1, 1, 1, 1, 1, 5, 6 & 1, 1, 1, 1, 2, 4, 5 \\
1, 1, 1, 1, 1, 1, 2, 9 & \quad 1, 1, 1, 1, 2, 2, 5 & 1, 1, 1, 1, 2, 4, 6 \\
1, 1, 1, 1, 1, 2, 5, 5 & \quad 1, 1, 1, 2, 2, 3, 5 & 1, 1, 2, 2, 2, 3, 5 \\
1, 1, 1, 1, 1, 3, 5 & \quad 1, 1, 1, 2, 2, 3, 6 & 1, 1, 2, 2, 3, 3, 3 \\
1, 1, 1, 1, 1, 3, 6 & \quad 1, 1, 1, 2, 2, 4, 4 & 1, 1, 2, 2, 3, 3, 4 \\
1, 1, 1, 1, 1, 3, 4, 4 & \quad 1, 1, 1, 2, 2, 4, 5 & 1, 1, 2, 3, 3, 3, 3 \\
1, 1, 1, 1, 1, 3, 4, 5 & \quad 1, 1, 1, 2, 3, 3, 5 & 1, 2, 2, 2, 3, 3 \\
1, 1, 1, 1, 1, 4, 4, 4 & \quad 1, 1, 2, 2, 2, 2, 5 & 1, 2, 2, 3, 3, 3 \\
1, 1, 1, 1, 2, 2, 2, 5 & \quad 1, 1, 1, 2, 3, 4, 4 & 1, 2, 2, 3, 3 \\
1, 1, 1, 2, 2, 2, 6 & \quad 1, 1, 2, 2, 2, 3, 3 & \\
1, 1, 1, 2, 2, 2, 7 & \quad 1, 1, 2, 2, 2, 3, 4 & \\
\end{align*}
\]

Of the 45 possible 9-tuples that might be the types of orthogonal designs we can only verify that two are! These two are

\[
\begin{align*}
1, 1, 1, 1, 1, 1, 1, 1 & \quad 1, 1, 2, 2, 2, 2, 2, 2, 2 \\
1, 1, 1, 1, 1, 1, 1, 2 & \quad 1, 1, 2, 2, 2, 2, 2, 2, 4 \\
1, 1, 1, 1, 1, 1, 2, 2 & \quad 1, 2, 2, 2, 2, 2, 2, 2, 2 \\
1, 1, 1, 1, 1, 2, 2, 2 & \quad 1, 2, 2, 2, 2, 2, 2, 3 \\
1, 1, 1, 1, 1, 2, 2, 2 & \quad 2, 2, 2, 2, 2, 2, 2, 2 \\
1, 1, 1, 1, 2, 2, 2, 2 & \quad 1, 2, 2, 2, 2, 2, 2, 2, 2 \\
\end{align*}
\]

\[n = 32\] While our methods give many designs in order 32 we have not yet calculated exactly which designs are missing as so much more work has yet to be done in deciding exactly what is happening in order 16.

\[n \equiv 0 \pmod{4}, n \neq 2^4\]

\[n = 12\] In this case \(\rho(n) = 4\), so we need only consider designs on \(\leq 4\) variables. We note that of all the numbers \(\leq 12\), seven is the only one which is not the sum of \(\leq 3\) squares.
one variable In [14] it was shown that a \( W(12, k) \) exists for every \( k \leq 12 \). Thus, all designs on one variable exist.

We next consider designs on four variables. There are 53 four-tuples \((s_1, s_2, s_3, s_4)\) where \( \Sigma s_i \leq 12 \), \( 1 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \). Of these, 32 are eliminated (by Proposition 24 and its corollary) as being the possible types of an orthogonal design of order 12. Of the 21 remaining we can construct the following by using Theorem 13:

\[
\begin{array}{cccccc}
\text{(type)} & 1\text{st row of } A & 1\text{st row of } B & 1\text{st row of } C & 1\text{st row of } D \\
(1, 1, 1, 1) & x_1 & 0 & 0 & x_2 & 0 & 0 & x_3 & 0 & x_4 & 0 & 0 \\
(1, 1, 1, 4) & x_3 & 0 & 0 & x_2 & 0 & 0 & x_3 & x_4 & x_5 & 0 & x_4 & x_5 \\
(1, 1, 1, 9) & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
(1, 1, 2, 2) & x_1 & 0 & x_2 & 0 & x_3 & 0 & x_4 & 0 & x_5 & 0 & x_6 & 0 \\
(1, 1, 2, 8) & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
(1, 1, 4, 4) & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
(1, 1, 5, 5) & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_3 & x_4 & x_5 \\
(1, 2, 2, 4) & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
(1, 2, 3, 6) & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_3 & x_4 & x_5 \\
(2, 2, 2, 2) & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
(2, 2, 4, 4) & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_3 & x_4 & x_5 \\
\end{array}
\]

In addition the Baumert-Hall array of order 12 gives type \((3, 3, 3, 3)\).

For the reader’s convenience we list those 4-tuples for which we have been unable to decide if an orthogonal design of that type exists.

\[
\begin{array}{cccc}
(1, 1, 1, 2) & (1, 1, 2, 3) & (2, 2, 2, 3) \\
(1, 1, 1, 3) & (1, 2, 2, 2) & (2, 2, 4) \\
(1, 1, 1, 8) & (1, 3, 3, 3) & (2, 2, 6) \\
\end{array}
\]

We now consider designs on three variables. There are 53 three-tuples \((s_1, s_2, s_3)\) where \( s_1 \leq s_2 \leq s_3 \) and \( \Sigma_{s_1} s_i \leq 12 \). We eliminate 16, because of Proposition 24 and its corollary. Many of the remaining 37 result from setting some of the variables in the designs on four variables we have found above equal to each other or to zero. In fact, the only orthogonal designs on three variables that we know exist arise in this fashion.

Again, for the reader’s convenience we list all those three-tuples which we cannot prove correspond to the type of an orthogonal design.

\[
\begin{array}{ccc}
(1, 1, 3) & (2, 2, 6) \\
(1, 3, 3) & (3, 3, 4) \\
(2, 2, 3) & \\
\end{array}
\]

There are 36 two-tuples \((s_1, s_2)\) where \( s_1 + s_2 \leq 12 \), \( 1 \leq s_1 \leq s_2 \). Of these, 3 cannot be the types of an orthogonal design, in view of Proposition 24 and its corollary.

We can easily obtain the remaining 33 by setting the variables in the designs above equal to each other or to 0. Thus, Proposition 24 and its
corollary give necessary and sufficient conditions for the existence of designs on two variables in order 12.

Although our methods give many designs in order 20 we have not made such extensive calculations there.

We would now like to briefly indicate how some of the results we have obtained can be used in studying weighing matrices. It was conjectured in [14] that if \( n \equiv 0 \pmod{4} \) then there is a \( W(n, k) \) for every \( k \leq n \). The first value of \( n \) for which this is not settled is \( n = 36 \). The missing weights there are \( k = 23, 24, 26, 28, 29, 30, 31 \).

We have constructed an orthogonal design of order 12 and type \((2, 5, 5)\) on the variables \( x_1, x_2, x_3 \). Let \( x_1 = I_3 \),

\[
x_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad x_3 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.
\]

The resulting matrix is a \( W(36, 26) \), as can be easily checked.

We have observed that there is an orthogonal design of order 12 and type \((6, 6)\). If in this design we let

\[
x_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}
\]

we obtain a \( W(36, 30) \). On the other hand, if we let

\[
x_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}
\]

we obtain a \( W(32, 24) \).

We have also found an orthogonal design of order 12 and type \((1, 5, 6)\) on the variables \( x_1, x_2, x_3 \). If we let

\[
x_1 = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},
\]

then it can be easily checked that the resulting matrix is a \( W(36, 29) \).

From the orthogonal design of order 12 and type \((2, 4, 6)\) on the variables \( x_1, x_2, x_3 \) we obtain a \( W(36, 28) \) by letting

\[
x_1 = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},
\]

Thus, the only \( W(36, k) \) yet to be found are for \( k = 23, 31 \).

We mention, without proof, that the circulant \( W(13, 9) \) that we have
found may be used with the results of [14] to obtain a $W(26, 18)$ and a $W(26, 20)$. These, in turn, will yield a $W(52, 36)$ and a $W(52, 43)$.

6. SOME OPEN QUESTIONS AND PROBLEMS

a) If $n = 4t$, $t$ odd, the only designs of type $(1, 1, 1, s)$ that we have found have $s = a^2$ from some integer $a$. We conjecture that this is always the case.

b) We suspect that if $n = 4t$, $t$ odd, then any design of type $(a, a, a, x)$ must have $(x/a) = s^2$, $s \in Q$.

c) If $n \equiv 0 \pmod{4}$, $n \neq 4t$, $t$ odd, $(t > 1)$, find some condition which says that there is a design on $\leq p(n)$ variables which is impossible to construct. (i.e. a theorem like our Proposition 25).

d) More modestly, show that one of the designs of order 16, that we have been unable to find, is impossible.

e) If $n = 4t$, $t$ odd, the only designs that we have been able to find of type $(1, 1, s)$ have $s = a^2 + b^2$. We conjecture that this is always the case. More generally, if $n = 4t$, $t$ odd then a design of type $(a, a, x)$ will exist if and only if $x/a = q_1^2 + q_2^2$, $q_i \in Q$.

f) Find families of "Williamson-type" matrices to use in these designs to construct new weighing matrices and new Hadamard matrices.

g) Show that the only $n \equiv 0 \pmod{4}$ for which every possible $p(n)$-tuple is the type of an orthogonal design is $n = 1, 2, 4, 8$. (Many of our previous questions would be answered by answering this one.)

h) We have shown that there is a $W(n, 4)$, for all $n \geq 10$ and a $W(n, 9)$ for all $n \geq 32$. Find the least integer $N$ such that a $W(n, 9)$ exists for all $n \geq N$. Obtain similar results for $W(n, s^2)$ by finding odd $n$'s for which $W(n, s^2)$ exists. As a first step, find a $W(31, 25)$.

i) Find other pairs of orthogonal designs of order 4 (or tuples of designs of order 8) which, pairwise, satisfy the matrix equation $XY^t = YX^t$ (see Construction 22)

Note added in proof

Since the submission of this paper there has been considerable progress on the problems mentioned in Section 6. The interested reader is referred to the following papers and their bibliographies.


References