Families of weighing matrices

Anthony V. Geramita, Norman J. Pullman,
and Jennifer S. Wallis

A weighing matrix is an $n \times n$ matrix $\mathbf{W} = W(n, k)$ with entries from \{0, 1, -1\}, satisfying $\mathbf{W}^T \mathbf{W} = k \mathbf{I}_n$. We shall call $k$ the degree of $\mathbf{W}$. It has been conjectured that if $n \equiv 0 \pmod{4}$ then there exist $n \times n$ weighing matrices of every degree $k \leq n$.

We prove the conjecture when $n$ is a power of 2. If $n$ is not a power of two we find an integer $t < n$ for which there are weighing matrices of every degree $\leq t$.

Taussky [1] suggested the following generalization of Hadamard matrices:

A weighing matrix is an $n \times n$ matrix $\mathbf{W} = W(n, k)$ with entries from \{0, 1, -1\}, satisfying $\mathbf{W}^T \mathbf{W} = k \mathbf{I}_n$. We shall call $k$ the degree of $\mathbf{W}$.

In [3, p. 433], it was conjectured that

(*) If $n \equiv 0 \pmod{4}$ then there exist $n \times n$ weighing matrices of every degree $k \leq n$.

(Note that an $n \times n$ weighing matrix of degree $n$ is an Hadamard matrix and so (*) is a generalization of the conjecture on the existence of Hadamard matrices of order $n$ for every $n \equiv 0 \pmod{4}$.)

In [4] the validity of (*) was established for $n \in \{4, 8, 12, 16, 20, 24, 28, 32, 40\}$ and partial results were obtained.

Received 9 October 1973. The work of the first two authors was supported in part by the National Research Council of Canada.
for \( n \in \{36, 44, 52, 56\} \) in that sets of values of \( k \) were obtained for which \( \bar{w}(n, k) \) exists.

For all \( n \) let \( g(n) \) be the maximum degree \( q \) for which there exist weighing matrices \( \bar{w}(n, k) \) for all degrees \( k \leq q \). Thus, conjecture (*) is equivalent to:

(*) \[ g(n) = n \text{ for all } n \equiv 0 \pmod{k}. \]

The methods of [2] can be used to show that \( g(2^n) \geq 34 \) for all \( n > 5 \). We show [Corollary 2 to our theorem] that in fact \( g(2^n) = 2^n \) for all \( n \) and hence establish (*) for all powers of 2. As another corollary to the theorem we show that \( g(2^k n) \geq 2^k \) for all odd \( n \) and all \( k \geq 1 \). This is better, asymptotically, than results obtained by the methods of [4].

Call \( \{M_1, M_2, \ldots, M_m\} \) an \( M \)-family of order \( n \) if for each \( i \), \( 1 \leq i \leq m \):

1. \( M_i \) is a weighing matrix of order \( n \) and degree \( i \), and
2. \( M_i M_m^t = M_m M_i^t \).

Let \( \mu(n) \) be the largest \( m \) for which an \( M \)-family of order \( n \) exists. Evidently \( g(n) \geq \mu(n) \).

**THEOREM.** If \( \mu(n) \geq m \) then \( \mu(2n) \geq 2m \).

Proof. Suppose \( \{M_1, M_2, \ldots, M_m\} \) is an \( M \)-family of order \( n \),

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \text{ and } I_p \text{ is the } p \times p \text{ identity matrix.} \]

Define

1. \( \bar{M}_i = I_2 \oplus M_i \) for each \( i \), \( 1 \leq i \leq m \),
2. \( \bar{M}_{m+i} = \bar{M}_i + A \oplus M_m \) for each \( i \), \( 1 \leq i \leq m-1 \), and
3. \( \bar{M}_{2m} = H \otimes M_m \).

It is easily verified that \( \{\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_{2m}\} \) is an \( M \)-family of order \( 2m \). The matrices defined in (a) and (c) satisfy (1) and (2) because the
$M^t_i$ do. The matrices defined in (b) satisfy (1) because the Hadamard product of $A$ and $I_2^t$ being the zero matrix implies they are (1, -1, 0)-matrices, and $M^t_{m+i}M^t_{m+i} = (m+i)I_{2n}$ because $A$ is skew symmetric; they satisfy (2) because $HA^t = AH$.

**Corollary 1.** \( u(2^k) = 2^k \) for all integers \( k \geq 1 \).

Proof. \( \{I_2^t, H\} \) is an $M$-family of order 2.

**Corollary 2.** \( g(2^k) = 2^k \) for all integers \( k \geq 1 \).

**Corollary 3.** (*) is true for all powers of 2.

**Corollary 4.** \( g(2^k n) \geq 2^k \) for all integers \( n \) and \( k \geq 1 \).

Proof. Each matrix \( I_n^t \oplus M^t_i \) is a weighing matrix of order \( nm \) and degree \( i \) if \( M^t_i \) is a weighing matrix of order \( n \) and degree \( i \).

Lemma 1 (1), 2 (1) and (iii) of [2] imply immediately that

(+) If (*) holds for \( n \) then \( g(2^t n) \geq n + 2t \) for all integers \( t \geq 0 \).

But Corollary 4 gives far better estimates of \( g(2^t n) \) than does (+) for all sufficiently large \( t \). For example, the results of [2] and (+) give us \( g(2^t 2k) \geq 2k + 2t \) but Corollary 4 gives us \( g(2^t 2k) \geq 2^{t+1} \) which is a better estimate for all \( t \geq 2 \).

**References**


[3] W.D. Wallis, Anne Penfold Street, Jennifer Seberry Wallis, 
Combinatorics: Room squares, sum-free sets, Hadamard matrices 
(Lecture Notes in Mathematics, 292. Springer-Verlag, Berlin, 

Department of Mathematics, 
Queen's University,  
Kingston, 
Ontario,  
Canada.