Combinatorial matrices

Jennifer Wallis

We investigate the existence of integer matrices $B$ satisfying the equation

$$BB^T = rI + sJ,$$

where $T$ denotes transpose, $r$ and $s$ are integers, $I$ is the identity matrix and $J$ is the matrix with every element $+1$.

Hadamard matrices are $(1, -1)$ matrices of order $n = 2^t$ which have $r = n$ and $s = 0$ in (1). We discuss equivalence of Hadamard matrices over the integers and show that all Hadamard matrices of order $4t$, where $t$ is odd and square-free are equivalent over the integers. Further, if $t$ is even and square-free and there is a Hadamard matrix of order $2t$, then there is a Hadamard matrix of order $4t$ which is equivalent over the integers to the diagonal matrix

$$\text{diag}(1, 2, \ldots, 2, 2m, \ldots, 2m, 4m).$$

$2m-1$ times $2m-1$ times

We now develop many methods for constructing Hadamard matrices. Many of these constructions use skew-Hadamard matrices, that is Hadamard matrices $H = I + R$ where $R^T = -R$, or $\sigma$-type matrices, that is $(1, -1)$ matrices $H = I + P$ of order $n$ where $P^T = P$ and $PP^T = (n-1)I$. We first develop some theory on the Willseon method for constructing skew-Hadamard matrices and show if $H$ is the order of a skew-Hadamard matrix (or type matrix) then there exists a skew-Hadamard (or type) matrix of order $(n-1)^d + 1$ where $\kappa = 2^{a+b+c+d}, b, c, d$ non-negative integers while $a$ is a positive (non-negative) integer.

The concept of supplementary difference sets, that is, a set of subsets such that when we take all the differences in each subset and collect them, each difference occurs a fixed number of times in the totality, is introduced and an example given. Hadamard designs on \( n \) distinct letters are shown to exist for \( n = 2, 4 \) and \( 8 \).

\((v, k, \lambda)\)-configurations are considered, that is, \((0, 1)\)-matrices \( B \) of order \( v \) such that \( v = k - \lambda \) and \( \lambda = \lambda \) in (1). We show two similar but distinct methods for proving there exists a \( \{q^2(q+2), q(q+1), q\} \) configuration whenever \( q \) is prime or \( q = 2^2, 2^3, 2^4, 3^2, 3^3 \) or \( 7^2 \). We prove that whenever a \((q, k, \lambda)\)-configuration exists, \( q \) a prime power, then a \( \{q(k^2+\lambda), qk, k^2+\lambda, k, \lambda\} \)-configuration exists.

We consider integer matrices satisfying

\[ BB^T = vI - J, \quad B^T J = 0 = JB \quad \text{and} \quad B^T = -B \]

and find that either the greatest common divisor of the elements of \( B \) is \( 1 \) or \( B \) has zero diagonal and \( +1 \) or \( -1 \) elsewhere. Also we show that if \( B \) is an integer matrix of order \( b \) satisfying

\[ BB^T = (p-q)I + qJ \]

\[ BJ = qJ \]

where \( p, q \) and \( d > 0 \) are constants then if \( z \), the least element of \( B \), satisfies

\[ z \leq \frac{d}{b} \quad \text{and} \quad z \leq \frac{d+p}{d+q} \]

where \( \omega \) is the greatest element of \( B \), then

\[ B = \frac{d}{b} J. \]

We give tables of the orders \( \leq 1000 \) of known Hadamard, skew-Hadamard and \( n \)-type matrices at the date of submission as well as lists of known classes of these matrices.