Integer matrices obeying generalized incidence equations

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We consider integer matrices obeying certain generalizations of the incidence equations for \((v, k, \lambda)\)-configurations and show that given certain other constraints, a constant multiple of the incidence matrix of a \((v, k, \lambda)\)-configuration may be identified as the solution of the equation.

We define \((v, k, \lambda)\)-configurations as usual (see [3]). If \(B\) is the \((0, 1)\) incidence matrix of a \((v, k, \lambda)\)-configuration and if \(A = bB\) where \(b\) is a positive integer, then

\[
\begin{align*}
AA^T &= b^2(k-\lambda)I + b^2\lambda J \\
A\overline{\lambda} &= b\lambda \\
\lambda \nu - 1 &= k(k-1),
\end{align*}
\]

with \(J\) as usual the matrix with every element \(+1\), and \(I\) the identity matrix. Ryser [2] proved a partial converse:

**LEMMA 1.** If \(A\) is a \(v \times v\) integer matrix satisfying equations (1) with \(b = 1\), then \(A\) is the incidence matrix of a \((v, k, \lambda)\)-configuration (and consequently has every entry \(0\) or \(1\)).

One might conjecture, in view of the powerful theorems of Ryser [2] and Bridges and Ryser [1], that an integer matrix satisfying (1) would necessarily be \(b\) times the incidence matrix of a \((v, k, \lambda)\)-configuration. But the matrix

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satisfies (1) with $b = 2$, $v = 7$, $k = 3$ and $\lambda = 1$. So we need other conditions on the matrix $A$ before we can ensure that every element is 0 or $b$. We shall prove:

**THEOREM 2.** If $A$ is a $v \times v$ matrix of non-negative integers which satisfies (1), and if every entry of $A$ is less than or equal to $b$, then $A$ is $b$ times the incidence matrix of a $(v, k, \lambda)$-configuration.

The corresponding result for non-positive $A$ and negative $b$ also holds.

By similar methods we shall obtain a result about more general equations:

**THEOREM 3.** Let $B$ be an integer matrix of order $v$ which satisfies

$$BB^T = (p-q)I + q\mathbf{w}$$

$$Bd = d\mathbf{w}$$

where $p$, $q$ and $d$ are constants and $d > 0$. Write $w$ and $z$ for the greatest and least elements of $B$ respectively, and $w = |v|$.

If

$$z \leq \frac{d}{v} = \delta \quad \text{and} \quad z \leq \frac{2\delta + p}{d + \delta w},$$

then $\delta$ is an integer, $p = d\delta = v\delta^2$, and $B = \delta B$.

1. Proof of Theorem 2

**LEMMA 4.** Let $B = \{b_{ij}\}$ of order $v$ be a matrix of non-negative integers such that

$$\sum_{j=1}^{v} b_{ij}^2 = p,$$

$p$ a constant, for every $i$, and let
\[ B^i = d^j, \quad d \text{ a non-zero constant. If } b^i_j = \frac{E}{d} \text{ for every } b^i_j, \text{ or if } b^i_j > \frac{E}{d} \text{ for every non-zero } b^i_j, \text{ then every entry of } B \text{ is } 0 \text{ or } \frac{E}{d}. \]

Proof. \[ \sum_{j=1}^{\nu} b^i_j \frac{E}{\nu} = p \quad \text{and} \quad \sum_{j=1}^{\nu} b^i_j = d, \quad \text{so} \]
\[ d \sum_{j} b^i_j \frac{E}{\nu} - p \sum_{j} b^i_j = dp - dp = 0; \]
that is
\[ \sum_{j} b^i_j (db^i_j - p) = 0. \]

From the data every term in this summation has the same sign, so every term is zero. So \[ b^i_j = 0 \text{ or } \frac{E}{d}. \]

COROLLARY 5. If there is a matrix \( B \) satisfying the conditions of Lemma 4, then \( d|p \).

Corresponding results may be obtained for matrices of non-positive integers.

Proof of Theorem 2. The matrix \( A \) satisfies the conditions of Lemma 4 with \( p = \frac{b^2k}{d} \) and \( d = bk \). So every entry is \( 0 \) or \( b \left( \frac{E}{d} \right) \).

Consider \( B = b^{-1}A \). \( B \) is an integer matrix satisfying Lemma 1, so it is the incidence matrix of a \((v, k, \lambda)\)-configuration, and we have the result.

2. Proof of Theorem 3

Proof of Theorem 3. Clearly \( p = \sum_{j} b^i_j \frac{E}{\nu} \) implies \( p \geq 0 \); and \( d > 0 \) implies \( p > 0 \). Consider the class of matrices
\[ C_\alpha = B + \alpha d \]
where \( \alpha \) is an integer and \( \alpha \geq \omega \). Every element of every member of this class is non-negative and
\[ C'_{a} = (p-q)I + (a^2 \nu + 2ad \nu + q)J \]
\[ C'_{a}J = (d+aw)J. \]

Then using Lemma 4, if every non-zero element of \( C_a \) is less than or equal to \( \beta \),
\[ \beta = a + \frac{ad+p}{d+aw}, \]
then every element is 0 or \( \beta \).

We show that the conditions on \( z \) imply that every element is \( \leq \beta \).

For
\[ z \leq \frac{w+\nu}{d+aw} \]
implies
\[ z(d+aw) \leq wd + p; \]
since \( z \leq \frac{d}{p} \) we have
\[ wd + \nu d + \nu \gamma u \leq wd + p \]
for any integer \( \gamma \geq 0 \), so
\[ z \leq \frac{(w+\nu)d+p}{d+(w+\nu)\nu}. \]
This means (putting \( \alpha = w + \gamma \)) that for any admissible \( \alpha \),
\[ z + \alpha \leq \alpha + \frac{ad+p}{d+aw}; \]
but \( z + \alpha \) is the greatest element of \( C_a \). Therefore, any element of \( C_a \) is 0 or \( \alpha + \frac{ad+p}{d+aw} \), so any element of \( B \) is \(-\alpha \) or \( \frac{ad+p}{d+aw} \).

Corollary 5 tells us that
\[ A(\gamma) = \frac{(w+\nu)d+p}{d+(w+\nu)\nu} - \frac{d+p(w+\nu)^{-1}}{d(w+\nu)^{-1}+p} \]
is integral for all integers \( \gamma \geq 0 \). Therefore \( \lim_{\gamma \to \infty} A(\gamma) \) must be an integer, so \( v|d \). Write \( d = v\delta \):
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\[ A(\gamma) = \frac{(\omega+\gamma)v\delta+p}{v\delta+(\omega+\gamma)v} \]

so \( v \mid p \). Write \( p = ev \):

\[ A(\gamma) = \frac{(\omega+\gamma)\delta+e}{\delta+(\omega+\gamma)} \]  

Choose \( n \) any integer greater than \( \delta + \omega \). Then

\[ A(n-\delta-\omega) = \frac{(n-\delta)\delta+e}{n} \]

so \( n \mid (e-\delta^2) \). But this is true for every large enough \( n \); hence \( e = \delta^2 \). That is

\[ d = v\delta \]
\[ p = v\delta^2 \]

so

\[ p = d\delta = v\delta^2 \]

Then we have

\[ \frac{\omega\delta+p}{d+au} = \frac{v\delta(n+\delta)}{v(\delta+a)} = \delta \]

for any \( \gamma \), so every element of \( B \) is \( -\alpha \) or \( \delta \). Now the row sum of \( B \)

is \( d = v\delta \) and the sum of the squares of the elements is

\[ p = v\delta^2 \];

together these imply

\[ B = \delta J \]

where \( \delta = \frac{d}{u} \).

References


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