Note

Amicable Hadamard Matrices

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Communicated by Marshall Hall, Jr.

Received August 24, 1970

If $X$ is a symmetric Hadamard matrix, $Y$ is a skew-Hadamard matrix, and $XY^T$ is symmetric, then $X$ and $Y$ are said to be amicable Hadamard matrices. A construction for amicable Hadamard matrices is given, and then amicable Hadamard matrices are used to generalize a construction for skew-Hadamard matrices.

We refer the reader to Marshall Hall, Jr. [1] and Jennifer Wallis [3] for the definitions of Hadamard matrix, skew-Hadamard matrix, skew-type, circulant and back-circulant. In [3] we define $m$-type matrices, which we will henceforth call amicable Hadamard matrices, to be a pair of Hadamard matrices $M$ and $N$ of the same order such that $M$ in skew-type, $N$ is symmetric and

$$MN^T = NM^T;$$

where the superscript $T$ denotes matrix transpose. Here we give another construction for amicable Hadamard matrices and generalize a theorem in [3].

We shall construct two Hadamard matrices of order $2y + 2$ of the form below with $X$ symmetric and $Y$ skew-type:

$$X = \begin{bmatrix}
1 & 1 & e & e \\
1 & -1 & -e & e \\
e^T & -e^T & A & -B \\
e^T & e^T & -B & -A
\end{bmatrix}, \quad Y = \begin{bmatrix}
1 & 1 & e & e \\
-1 & 1 & e & -e \\
-e^T & -e^T & C & D \\
e^T & e^T & -D & C
\end{bmatrix},$$

where $A, B, C, D$ are of order $y$, $A, B, D$ are symmetric, and $C$ is skew-type, and $e = [1, ..., 1]$ is $1 \times y$. The Hadamardness of $X$ and $Y$ imposes the following properties on the submatrices of $X$ and $Y$:

$$eA^T = e = eB^T, \quad AB^T = BA^T, \quad eC^T = e = eD^T, \quad CD^T = DC^T.$$
Then $XY^T$ is symmetric if and only if

$$ACT - BD^T, BC^T + AD^T$$

are symmetric.

We recall lemma 6 of [3]:

**Lemma 1.** If $P$ is circulant and $Q$ is back-circulant then $PQ^T$ is symmetric.

Let $R = (r_{ij})$ of order $y$ be defined by $r_{i,y-i+1} = 1$ and for $j \neq y - i + 1$, $r_{ij} = 0$. Then if $A, B, D$ are back-circulant matrices, with $AR, BR$ and $DR$ circulant and symmetric, and if $C$ is circulant such that $X$ is a symmetric Hadamard matrix, and $Y$ is a skew-Hadamard matrix, then $X$ and $Y$ are amicable Hadamard matrices.

Let $y$ be prime. Define $W = (w_{ij})$ by $w_{ij} = 0$, $w_{ij} = \chi(j - i)$ for $j \neq i$ where $\chi(b)$ is the Legendre symbol. For $y$ (prime) $\equiv 1 \pmod{4}$ $W^T = W$.

Now choose $A = (I + W)R$ and $B = (I - W)R$.

In his paper [2] G. Szekeres shows how to construct twin difference sets which will yield the required $C$ and $D$ for $q$, where $(2q + 1)$ (prime power) $\equiv 3 \pmod{4}$. Also with $H_0$, $i = 0, 1, 2, 3$, as in the proof of theorem 5 of [2] $K = H_0 \cup H_3$ and $K = H_0 \cup H_5$ can be used to form the required $C$ and $D$ for $y = 5, 13, 29, 53$. So we have

**Theorem 2.** If $q$ is a prime such that

(i) $5, 13, 29, 53, or$

(ii) $2q + 1$ is a prime, and $q$ is odd,

then there are amicable Hadamard matrices of order $2(q + 1)$.

Summarizing, using the proof of Lemma 8 of [3], we have amicable Hadamard matrices of the following orders:

| I | 5; |
| II | $p^r + 1$ where $p^r$ (prime power) $\equiv 3 \pmod{4}$; |
| III | $2(q + 1)$ where $q$ (prime) $\equiv 1 \pmod{4}$ and $2q - 1$ is prime; |
| IV | $S$, where $S$ is a product of any of the above orders. |

We note the following theorem, which is a generalization of corollary 9 of [3]. The proof is similar to that in [3].

**Theorem 3.** Let $m$ and $m'$ be the orders of amicable Hadamard matrices. If there is a skew-Hadamard matrix of order

$$\frac{(m - 1)m'}{m}, \quad \frac{(m - 1)(m' - 4)}{m}$$

then...
then there is a skew-Hadamard matrix of order

(i) \( m'(m' - 1)(m - 1) \),  \( m'(m' - 1)(m' - 4)(m - 1) \),

respectively.

REFERENCES