A CLASS OF GROUP DIVISIBLE DESIGNS

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Let \( Q \) be the incidence matrix of a cyclic projective plane of order \( q \), which exists whenever \( q \) is a prime power, then

\[
QQ^T = qI + J, \quad QJ = (q+1)J.
\]

Let \( Q = \sum_{d \in D} T^d \) where \( T \) is the matrix which is 1 in the \((i,i+1)\) position (reduced mod \( q^2+q+1 \)) for \( i = 1, \ldots, q^2+q+1 \). Then

\[
Q^2 = \sum_{d \in D} T^{2d} + 2 \sum_{a, f \in D} T^{a+f}.
\]

It is shown in Cermenta and Seberry [3, Theorem 4.124] that \( Q^2 \) has entries 0,1,2. Let \( C = \sum_{d \in D} T^{2d} \). Then \( D \) is a difference set.

\( C \) satisfies

\[
CC^T = qI + J, \quad CJ = (q+1)J.
\]

Let \( W = Q^2 - J \) which has entries \((0,1,-1)\). Let \( A \) be the matrix which is 1 where \( W \) is 1 and zero elsewhere. Let \( B \) be the matrix which is 1 where \( W \) is -1 and zero elsewhere. Then

\[
W = A - B \quad \text{and} \quad A + B = J - C.
\]

Now \( W \) and \( A + B \) satisfy

\[
WW^T = AA^T + BB^T - AB^T - BA^T = q^2 I,
\]

\[
(A+B)(A+B)^T = AA^T + BB^T + AB^T + BA^T = qI + (q^2-q)J.
\]

Hence

\[
AA^T + BB^T = \frac{1}{2}(q^2+q)I + \frac{1}{2}(q^2-q)J,
\]

\[
AB^T + BA^T = \frac{1}{2}(q^2-q^2)I + \frac{1}{2}(q^2-q)J.
\]

So

\[
\begin{pmatrix}
A & B \\
B & A
\end{pmatrix}
\]

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is a \((0,1)\) matrix with \(q^2\) elements in each row and column with inner product between any two rows 0 (if they are in different submatrices of the partition) or \(\frac{1}{2}(q^2-q)\) (otherwise).

We refer the reader to [1,2] for the definition and some important properties of symmetric regular group divisible designs. Now we can say:

Suppose \(q\) is a prime power; then there is a symmetric regular group divisible design with parameters

\[(v,b,r,k,\lambda_1,\lambda_2,\mu,n) = (2(q^2+q+1), 2(q^2+q+1), q^2, q^2, \frac{1}{2}(q^2-q), q^2+q+1, 2)\].

REFERENCES


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