An Infinite Family of Quasi Orthogonal Matrices with Two Levels Constructed via SBIBD

N. A. Balonin* and Jennifer Seberry†

October 4, 2014

We show that if \( B \) is the incidence matrix of an SBIBD(\( v, k, \lambda \)), then there exists a two-level quasi-orthogonal matrix, \( S \), satisfying

\[
S^\top S = SS^\top = (ka^2 + (v - k)b^2)I_v = cI_v,
\]

where \( c = ka^2 + (v - k)b^2 \) is a constant and \( I_v \) is the identity matrix of order \( v \). To ensure the matrix \( S \) is quasi-orthogonal we must have

\[
\lambda a^2 - 2(k - \lambda)ab + (v - 2k + \lambda)b^2 = 0.
\]

We apply this result to families of SBIBD including the projective planes, hyperplanes, Hadamard cores and Menon difference sets (regular Hadamard matrices), obtaining a new infinite family.

Keywords: Hadamard matrices; quasi-orthogonal matrices; symmetric balanced incomplete block designs (SBIBD); regular Hadamard matrices; difference sets; Hadamard difference sets; Menon difference sets; Singer difference sets; 05B20.

1 Introduction

SBIBDs are of considerable use and interest to image processing (compression, masking) and to statisticians undertaking medical or agricultural research.

We note that the strict definition of an orthogonal matrix, \( X \), of order \( n \), is that \( X^\top X = XX^\top = I_n \) where \( I_n \) is the identity matrix of order \( n \). In this paper we consider \( X^\top X = XX^\top = cI_n \) where \( c \) is a constant. We call these quasi-orthogonal matrices \([4]\) to avoid any confusion.

*Saint Petersburg State University of Aerospace Instrumentation, 67, B. Morskaia St., 190000, St. Petersburg, Russian Federation. Email: korbendfs@mail.ru
†Centre for Computer and Information Security Research, School of Computer Science and Software Engineering, EIS, University of Wollongong, NSW 2522, Australia. Email: jennifer.seberry@uow.edu.au
2 Definitions

This paper studies the construction of some quasi-orthogonal matrices.

**Definition 1.** For the purposes of this paper we will consider an $SBIBD(v,k,\lambda)$, $S$, to be a $v \times v$ matrix, with entries 0 and 1 with $k$ ones per row and column, and the inner product of distinct rows and/or columns to be $\lambda$. For these matrices $\lambda(v-1) = k(k-1)$.

We note that for every $SBIBD(v,k,\lambda)$ there is a complementary $SBIBD(v,v-k,v-2k+\lambda)$. One can be made from the other by interchanging the 0’s of one with the 1’s of the other.

**Definition 2.** A real square matrix $S = (s_{ij})$ of order $n$ is called quasi-orthogonal if it satisfies $S^T S = SS^T = cI_n$, where $I_n$ is the $n \times n$ identity matrix, and $c$ is a constant real number.

In this and future work we will only use quasi-orthogonal to refer to matrices with modulus of real elements $\leq 1$ [4], where at least one entry in each row and column must be 1. Hadamard matrices [7], symmetric conference matrices [3], and weighing matrices [11] are the best known of these matrices with entries from the unit disk [10].

**Definition 3.** The values of the entries of the quasi-orthogonal matrix, $S$, are called levels.

Hadamard matrices are two-level matrices and symmetric conference matrices and weighing matrices are three-level matrices. Quasi-orthogonal matrices with maximal determinant of odd orders have been discovered with a larger number of levels [2].

3 Constructions for Two-Level Quasi-Orthogonal Matrices

We now use $SBIBDs$ to construct two-level quasi-orthogonal $SBIBDs$.

**Theorem 1.** Let $S$ be made from an $SBIBD(v,k,\lambda)$, $B$, by replacing the 1’s by $a$ and the 0’s by $-b$. Then $S$ is a two-level quasi-orthogonal $SBIBD$ when $S$ satisfies

$$S^T S = SS^T = cI_v$$

where $c = ka^2 + (v-k)b^2$ is a constant, the weight of $S$, and $I_v$ is the identity matrix of order $v$. The characteristic equation

$$\lambda a^2 - 2(k-\lambda)ab + (v-2k+\lambda)b^2 = 0,$$

(2)

gives two solutions,

$$b = \frac{\lambda}{(k-\lambda)\pm \sqrt{k-\lambda}} a, \quad c = ka^2 + (v-k)b^2.$$

$a$ and $b$ are the levels. If $b > a$ we have to choose the second level to be $1/b$, to ensure entries are from the unit disk for the complementary $SBIBD$. The determinant is $c^2$. 

Proof. $S$, satisfying equations (1) and (2) is a two-level quasi-orthogonal matrix by definition.

Remark 1. As noted above for every $SBIBD(v, k, \lambda)$ there is a complementary $SBIBD(v, v-k, v-2k+\lambda)$. The characteristic equation of the $SBIBD(v, k, \lambda)$ is equation 2:

$$\lambda a^2 - 2(k - \lambda)ab + (v - 2k + \lambda)b^2 = 0.$$

The characteristic equation of the $SBIBD(v, v-k, v-2k+\lambda)$ is the equation:

$$(v - 2k + \lambda)a^2 - 2(k - \lambda)ab + \lambda b^2 = 0.$$

Hence the solutions of the characteristic equation for the $SBIBD(v, k, \lambda)$ and the $SBIBD(v, v-k, v-2k+\lambda)$ are inverses of each other. Thus if $b > a$ for any solution, using the inverse solution from the complementary design will give a solution with $a > b$ by mapping $b$ to $a$ and $a$ to $1/b$.

We will say, that an attractor of a solution of an $SBIBD(v, k, \lambda)$ is a Hadamard matrix, if $b$ grows to 1 with growing $v$. If $b$ has the opposite action, falling and approaching 0, we will say, that the attractor is a weighing matrix. To construct solutions of the first type, let us choose the $SBIBD$, $B$, to be that with the bigger number of 1’s per row and column, i.e., $v < 2k$.

Corollary 1 (From Hadamard Matrices). Suppose there exists an Hadamard matrix of order $4t$, then there exists an $SBIBD(4t-1, 2t, t)$. Hence we have a two-level quasi-orthogonal matrix, $S$, satisfying equations (1) and (2) with levels $a$ and $b$, for $b = \frac{t}{t+\sqrt{t}}a$, $\det(S) = c^{\frac{4t}{t+1}}$.

For the “+” sign, choosing $a = 1$, we have the first solution

$$b = \frac{t}{t + \sqrt{t}}, \quad c = 2t + (2t-1)b^2 = 4t + \sqrt{t},$$

For the “−” sign, choosing $a = 1$ and inverted level $1/b$, we have

$$b = \frac{t - \sqrt{t}}{t}, \quad c = 2t - 1 + 2b^2,$$

for the complementary $SBIBD(4t-1, 2t-1, t-1)$ with smaller $b$ and smaller determinant.

Let us take as a principal solution the matrix with maximal determinant, the complementary solution has the same structure: it is an analogue of the equivalent version for an Hadamard matrix. The first (extremal) solution has been called a Mersenne matrix [6]. The conditions for existence of these matrices for all orders $4t - 1$ as been observed in [1], [5].

Example 1. For $SBIBD(7, 4, 2)$ consider $\text{circ}(a, a, a, -b, a, -b, -b)$ with characteristic equation $a^2 - 4ab + 2b^2 = 0$, $\det(S) = c^2$. The principal solution, see Fig. 1a (entries $a$ and $-b$ given as white and black squares), has
\[ a = 1, \quad b = \frac{2}{2 + \sqrt{2}} = 2 - \sqrt{2}, \quad c = 4a^2 + 3b^2 = 22 - 12\sqrt{2} = 5.0294. \]

The second solution for complementary SBIBD, \( \text{circ}(-b, -b, -b, a, -b, a, a) \), see Fig. 1b (smaller than above values \( b \), here and later, given as red squares), has
\[ a = 1, \quad b = \frac{2 - \sqrt{2}}{2}, \quad c = 3a^2 + 4b^2 = 9 - 4\sqrt{2} = 3.3431. \]

with \( b \) half the size. The two determinants are \( 2.8531 \times 10^2 \) and \( 6.8319 \times 10 \) respectively.

We note that there exist SBIBD\((v,k,\lambda)\) for \( v = 4t + 1, 4t, 4t - 1, 4t - 2 \) see the La Jolla Difference Set Repository [8] for many parameter sets which can make circulant matrix SBIBDs from difference sets.

![Figure 1: Quasi-orthogonal matrices for order 7: Mersenne Family](image)

To construct solutions of the second type (where the weighing matrix is an attractor), let us choose the SBIBD, \( B \), to be that with the smaller number of 1’s per row and column, i.e., \( v > 2k \).

**Corollary 2** (Singer Difference Sets and Projective Planes). Let \( p \neq 2 \) be a prime power. Then there exists an SBIBD\((p^2 + p + 1, p + 1, 1)\). Hence we have a two-level quasi-orthogonal matrix, \( S \), satisfying equations (1) and (2) for
\[ b = \frac{1}{p \pm \sqrt{p}}a, \quad c = (p + 1)a^2 + p^2b^2, \quad \det(S) = \frac{c^{\frac{p^2+1}{2}}}{2}, \]

For \( p = 2 \) we have two cases: the “+ sign” corresponds to the principal solution, given above; the “− sign” corresponds to the complementary SBIBD\((p^2 + p + 1, p^2, p^2 - p)\).

These two-level quasi-orthogonal matrices we will call the Singer family.
Example 2. Consider $p = 3$ which gives an $SBIBD(13, 4, 1)$. We use \( \text{circ}(-b, a, -b, a, a, -b, -b, a, -b, -b, -b) \) to generate this matrix, with characteristic equation $a^2 - 6ab + 6b^2 = 0$. There are two solutions, see Fig. 2a where $b = \frac{3 \pm \sqrt{3}}{6}$, and Fig. 2b where $b = \frac{3 - \sqrt{3}}{6}$, so we have

$$
a = 1, \quad b = \frac{1}{3 + \sqrt{3}} = \frac{3 \pm \sqrt{3}}{6}, \quad c = 4a^2 + 9b^2 = 7 \pm \frac{3\sqrt{3}}{2}, \quad = 9.5981 \text{ or } 4.4019
$$

for the required two-level quasi-orthogonal matrices. They have

$$
\text{det}(S) = \left(7 \pm \frac{3\sqrt{3}}{2} \right)^{\frac{13}{2}}
$$

$$= 2.4221 \times 10^6 \text{ or } 1.5264 \times 10^4.
$$

\[\square\]

Figure 2: Quasi-orthogonal matrices for order 13: continuation of Singer Family

The first two Singer matrices of orders $v = 3 \ (p = 1, \text{"+ sign"})$, $7 \ (p = 2, \text{"- sign"})$ are members of the Mersenne family. The third order $v = 13 \ (p = 3, \text{both signs})$ belongs to the set of numbers $4k + 1$, $k = 3$. However they are different from the orders $4k + 1$, $k$ is 1 or even, of the three-level matrices of the Fermat family [6]. This means that the Singer family is similar to the Mersenne matrices in structure.

Corollary 3 (Hyper Planes). Let $p \neq 2$ be a prime power. Then there exists an $SBIBD(\frac{p^n - 1}{p - 1}, \frac{p^{n-1} - 1}{p - 1}, \frac{p^{n-2} - 1}{p - 1})$. Hence we have a two-level quasi-orthogonal matrix, $S$, satisfying equations (1) and (2) for

$$
b = \frac{p^{n-2} - 1}{(p^{n-2} \pm \frac{p^{n-2}}{p^n})(p - 1)} a.
$$
For \( p = 2 \) we have two cases: the “+ sign” corresponds to the principal solution, given above, the “− sign” corresponds to the complementary \( SBIBD(\frac{p^{n-1}}{p-1}, p^{n-1} - \frac{p^{n-1} - 1}{p-1} + 1) \), for \( n = 3 \) it gives \( v = 7, k = 4, \lambda = 2 \) and \( a = 1, b = 2 - \sqrt{2} \), see Fig 1a.

We note the dual appearance of some of the matrices of this family with two-level Mersenne and Singer matrices, as discussed above.

**Corollary 4** (Menon Sets and Regular Hadamard Matrices). Suppose there exists a regular Hadamard matrix of order \( 4m^2 \), then there exists an \( SBIBD(4m^2, 2m^2 - m, m^2 - m) \). Hence we have two two-level quasi-orthogonal matrix, \( S \), satisfying equations (1) and (2) for \( b = \frac{m-1}{m+1}a \). The principal solution well known as a regular Hadamard matrix with

\[
a = 1, b = 1, c = 4m^2 \text{ has } \det(S) = (4m^2)^{2m^2}.
\]

The second solution gives

\[
b = \frac{m-1}{m+1}, \quad c = \frac{4m^4}{(m+1)^2}, \quad \det(S) = \left( \frac{4m^4}{(m+1)^2} \right)^{2m^2}
\]

for the two-level quasi-orthogonal matrix with smaller determinant.

**Example 3.** For \( SBIBD(16,6,2) \) we have two levels \( a \) and \( -b \), see Fig. 3a; and \( S^t S = SS^t = (6a^2 + 10b^2)I_{16} = cI_{16} \) with \( a = 1, b = 1; \) \( c = 16 \) and \( \det(S) = 16^8 \approx 4.2950 \times 10^9 \). The second solution, see Fig. 3b, has \( a = 1, b = \frac{1}{3}, \) and \( c = 7 \frac{1}{3} \) and \( \det(S) = 6.5388 \times 10^6 \).

![Figure 3: Regular Hadamard and quasi-orthogonal matrices for order 16](image)

Replacing the 1’s by 1 and the -1’s by 0 gives the \( SBIBD(16,6,2) \). Replacing the 1’s by \( a \) and the -1’s by \( b \) gives the two-level quasi-orthogonal matrix required.
4 Conclusions

We note the dual existence of some matrices of Singer family with two-level matrices of the Mersenne family, which are defined for orders $4k - 1$, where $k$ is integer [6]. All other useful references to this question may be found in [4].

Matrices of the Singer family also have orders belonging to the set of numbers $4k + 1$, where $k$ is odd: these are different from the three-level matrices of Fermat family [6] with orders $4k + 1$, $k$ is 1 or even. The latter exist for orders $v$, where $v - 1$ is order of a regular Hadamard matrix, described above via Menon sets.

Orders $4k + 1$, $k$ is odd, belong to non-ordinary Fermat matrices. Examples of regular Hadamard matrices of order 36, giving the first non-ordinary Fermat matrix of order 37, have been placed at site [9]. They use regular Hadamard matrices as a core and have the same (as the ordinary ones) level functions. We call them Balonin-Seberry-Fermat matrices and we will study them more fully in our future work.

The main conclusion (about alternating matrices) follows: orders $4k + 1$, $k$ odd, belong to alternating two- and three-level matrices of the Singer and Fermat families. The Singer and Mersenne families have matrices in common, they, and Hyper plane SBIBD based matrices, are similar to each-other and distinct from any three level Fermat family.

5 Acknowledgements

The authors would like to acknowledge Professor Mikhail Sergeev for his advice regarding the content of this paper. The authors also wish to sincerely thank Tamara Balonina for converting this paper into printing format. We acknowledge the use of the http://www.mathscinet.ru and http://www.wolframalpha.com sites for the number and symbol calculations in this paper.

References


