Contemporary Mathematics

A Journey of Discovery: Orthogonal Matrices and Wireless Communications

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Dedicated to Joseph Gallian on his 65th birthday and the 30th anniversary of his Duluth REU

Abstract. Real orthogonal designs were first introduced in the 1970's, followed shortly by the introduction of complex orthogonal designs. These designs can be described simply as square matrices whose columns are formally orthogonal, and their existence criteria depend on number theoretic results from the turn of the century. In 1999, generalizations of these designs were applied in the development of successful wireless communication systems, renewing interest in the theory of these orthogonal designs. This area of study represents a beautiful marriage of classical mathematics and modern engineering.

This paper has two main goals. First, we provide a brief and accessible introduction to orthogonal design theory and related wireless communication systems. We include neither the mathematical proofs of the relevant results nor the technical implementation details, rather hoping that this gentle introduction will whet the reader's appetite for further study of the relevant mathematics, the relevant engineering implementations, or, in the best case scenario, both. Second, in light of the dedication of this paper to Joe Gallian, who was an extraordinary undergraduate research advisor to so many of us and who inspired so many of us to become undergraduate research advisors ourselves, we describe the involvement of undergraduates in research involving these orthogonal designs and related communications systems.

1. Introduction and Motivation

Three accepted facts regarding wireless communications systems are that bandwidth is scarce, that multipath fading must be combated, and that wireless communications systems have grown into extremely complex systems made of many interacting components [**24**]. *Space-time block codes* (defined formally in Section 2) address each of these issues by elegantly combining two forms of diversity: space diversity and time diversity. This combination of diversity was a major step in moving the capacity of wireless communications systems towards the theoretical limits.

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A certain class of space-time block codes, known as complex orthogonal space-time block codes, are built using mathematical constructs known as generalized complex orthogonal designs. Orthogonal designs were first introduced in the mid-1970's [**10, 12, 13, 14**], however, mathematical results in other contexts dating from the 1890's laid the foundation for these combinatorial structures [**1, 2, 3, 16, 17, 23**]. The application of orthogonal designs and their generalizations as as complex orthogonal space-time block codes has been successful. For example, these codes are used in the Third Generation Mobile Standard and the proposed standard for wireless LANs IEEE 802.11n.

Our introductory overview of complex orthogonal space-time block codes does not attempt to match the technical rigor or completeness of Calderbank and Naguib's survey on this topic [**7**]. We refer the reader to their survey, as well as to the other articles in the bibliography, for a more advanced treatment of the relevant mathematics and implementation details. The books by Jafarkhani [**18**] and Larsson and Stoica [**20**] provide particularly comprehensive treatments of the topic.

In Section 2, we provide the necessary definitions and background information. In Section 3, we discuss the two main research problems in this area, including a narrative journey of undergraduate involvement in the solution of one of these main problems. Section 4 provides a brief overview of other related research problems. The paper is concluded in Section 5.

2. Definitions and Preliminaries

In modern wireless communication systems, data are sent from one or more transmit antennas to one or more receive antennas. In this paper, we consider one model for sending data over fading channels using multiple transmit antennas and one or more receive antennas. The motivation for using multiple transmit antennas is that multiple copies of data sent from different physical positions may be corrupted in different ways during transmission, which may better allow the receiver to recover the intended data. In other words, through appropriate signal processing at the receiver, the independent paths provided by the multiple transmit antennas can be considered as one channel that is more reliable than any of the single independent paths. This idea is known as space diversity. The model we consider also sends multiple copies of data at different timesteps, again increasing the likelihood that the receiver can recover the intended data. This idea is known as time diversity. Space and time diversity are explained in more depth in several of our references, e.g., [**7, 18, 20**]. In the model that we consider, the data (or signals) sent between antennas are modelled using complex variables.

In the most general form, a *space-time block code* for n transmit antennas is a mapping from k complex variables $\{z_1, \ldots, z_k\}$ onto an $r \times n$ matrix $\mathbf{G}$, wherein each of the n columns of $\mathbf{G}$ represents the transmissions of a distinct antenna and each of the rows represents the transmissions at a given timestep. The entries in the matrix determine which antenna should send which signal at which time, so that an entry of z_l in position (i, j) indicates that the signal corresponding to the complex variable z_l should be transmitted by the j^{th} antenna in the i^{th} timestep. An entry of 0 indicates that the corresponding antenna does not transmit during the corresponding timestep. As the number of rows r represents the number of timesteps required to send the given data, and as this model requires us to receive all of the data before we can decode, r is referred to as the *decoding delay* of the code. The ratio k/r of number of distinct variables to the number of rows is a measure of the code's efficiency, and this is referred to as the *rate* of the code.

Complex orthogonal space-time block codes (COSTBCs) require that the matrix $\mathbf{G}$ is a *generalized complex orthogonal design*, the formal definition of which we will develop below. Qualitatively, COSTBCs require that the columns of the matrix $\mathbf{G}$ are orthogonal, and it is this orthogonality (in part) that makes COSTBCs attractive in practice: The orthogonality permits a simple maximum-likelihood decoding algorithm which, through only linear combining at the receiver, decouples the signals transmitted from the multiple antennas [**29**]. These codes also achieve full diversity [**29**].

Real orthogonal designs were first defined and studied by Geramita, Geramita, and Seberry Wallis [**10, 12, 13, 14**]:

DEFINITION 2.1. A *real orthogonal design* of order n and type $(s_1, s_2, \ldots, s_k)$ ($s_l > 0$) in real commuting variables $x_1, x_2, \ldots, x_k$, is an $n \times n$ matrix $\mathbf{A}$ with entries from the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_k\}$ satisfying

$$\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A} = \sum_{l=1}^{k} s_l x_l^2 \mathbf{I}_n,$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix.

The generalization to the complex domain followed shortly thereafter [**9**]:

DEFINITION 2.2. A *complex orthogonal design* of order n and type $(s_1, s_2, \ldots, s_k)$ ($s_l > 0$) in real commuting variables $x_1, x_2, \ldots, x_k$, is an $n \times n$ matrix $\mathbf{C}$ with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_k, \pm ix_1, \pm ix_2, \ldots, \pm ix_k\}$ satisfying

$$\mathbf{C}^H\mathbf{C} = \sum_{l=1}^{k} s_l x_l^2 \mathbf{I}_n,$$

where H is the Hermitian transpose (the transpose complex conjugate) and $i^2 = -1$.

An alternative definition for a complex orthogonal design $\mathbf{C}$ has entries from the set $\{0, \pm z_1, \ldots, \pm z_k, \pm z_1^*, \ldots, \pm z_k^*\}$, where the z_l are complex commuting variables and z_l^* denotes the complex conjugate of z_l, such that

$$\mathbf{C}^H\mathbf{C} = \sum_{l=1}^{k} s_l |z_l|^2 \mathbf{I}_n.$$

This latter definition is commonly used in the signal processing literature for its application to space-time block codes [**21, 22, 29**].

EXAMPLE 2.3. The following matrix $\mathbf{C}$ is a complex orthogonal design on complex variables z_1 and z_2, using the latter definition above:

$$\mathbf{C} = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}$$

In fact, this simple complex orthogonal design was the first to be employed as a space-time block code: The application was presented by Alamouti in 1998 [**5**]. (Alamouti himself did not refer to his code as a space-time block code, as that term was coined later by Tarokh, Jafarkhani, and Calderbank [**29**].) Alamouti [**5**] presented a very simple maximum-likelihood decoding scheme for this code, later reviewed and generalized by Tarokh et al. [**29**], that hinges on the orthogonality of the columns of $\mathbf{C}$.

Classical results from the turn of the century [**1, 2, 3, 16, 17, 23**] have been shown to imply that real and complex orthogonal designs exist only for limited values of n. The mathematics literature refers to the existence problem for orthogonal designs as the Hurwitz-Radon problem [**11**], and the results utilize the Hurwitz-Radon function [**26**], also referred to in the mathematics literature as the Radon-Hurwitz numbers [**2, 6**] or the Hurwitz-Radon-Eckmann formula [**8, 25**]. We recall the following definitions:

Definition 2.4. A set of $n \times n$ real matrices $\{\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_m\}$ is called a size m *Hurwitz-Radon family of matrices* if $\mathbf{B}_i^T \mathbf{B}_i = \mathbf{I}_n$ and $\mathbf{B}_i^T = -\mathbf{B}_i$ for all $1 \leq i \leq m$ and if $\mathbf{B}_i \mathbf{B}_j = -\mathbf{B}_j \mathbf{B}_i$ for all $1 \leq i < j \leq m$.

Definition 2.5. Let $n = 2^a b$, b odd, and write $a = 4c + d$ where $0 \leq d < 4$. *Radon's function* is the arithmetic function $\rho(n) = 8c + 2^d$.

Radon showed that a Hurwitz-Radon family of $n \times n$ matrices contains strictly less than $\rho(n)$ matrices, with the bound of $n-1$ matrices being achieved if and only if n = 2, 4, or 8 [**23**]. Tarokh et al. provide the details showing the equivalence of Radon's result to the fact that real orthogonal designs of order n exist only for n = 2, 4, or 8 [**29**], assuming all entries in the design are nonzero. An extension of this result shows that complex orthogonal designs of order n exist only for $n = 2$, assuming all entries in the design are nonzero [**29**]. Liang also provides a detailed summary of Hurwitz-Radon theory in the context of his treatment of real orthogonal designs [**21**].

Another connection between real orthogonal designs and Hurwitz-Radon theory was provided by Geramita, Geramita, and Seberry Wallis in their answer to the fundamental question concerning the maximum number of variables permissable in an $n \times n$ real orthogonal design. In particular, they showed that the number of variables in an $n \times n$ real orthogonal design is less than or equal to $\rho(n)$ [**10**]. Additionally, Calderbank and Naguib summarize certain connections among the rate of real orthogonal designs, Hurwitz-Radon theory, amicable orthogonal designs, and Clifford algebras [**7**].

Tarokh et al. recognized that Alamouti's successful coding scheme based on the matrix $\mathbf{C}$ given in Example 2.3 could be generalized by employing more general matrices with mutually orthogonal columns. Due to the severe limitations on the sizes of orthogonal designs and upon realizing the potential for utilizing orthogonal designs in wireless communications systems, Tarokh et al. [**29**] defined generalized, or rectangular, complex orthogonal designs as follows:

Definition 2.6. A *generalized complex orthogonal design (GCOD)* of order n is an $r \times n$ matrix $\mathbf{G}$ with entries from $\{0, \pm z_1, \dots, \pm z_k, \pm z_1^*, \dots, \pm z_k^*\}$, or products of these complex indeterminants with the imaginary unit i, such that

$$\mathbf{G}^H \mathbf{G} = \sum_{l=1}^{k} |z_l|^2 \mathbf{I}_n.$$

If the entries of $\mathbf{G}$ are allowed to be complex linear combinations of the complex variables and their conjugates, then the design $\mathbf{G}$ is called a *generalized complex linear processing orthogonal design*.

Using a well-known normalization result [**29**], it is assumed throughout that each column of a generalized complex orthogonal design includes exactly one position occupied by $\pm z_l$ or $\pm z_l^*$, for each $1 \le l \le k$ and each row includes at most one position occupied by $\pm z_l$ or $\pm z_l^*$, for each $1 \le l \le k$. Geramita and Seberry provide a comprehensive reference on orthogonal designs [**11**], and Liang reviews and defines all of the relevant generalizations [**22**].

By extending the definition of a complex orthogonal design from the square case to the rectangular case, Tarokh et al. were able to consider orthogonal matrices for any numbers of columns. In turn, this allowed the development of a theory for complex orthogonal space-time block codes for any number of antennas [**29**]. As with the Alamouti code [**5**], the columns of these rectangular designs are orthogonal, and hence the associated codes enjoy a simple decoding algorithm [**7, 29**]. Furthermore, these rectangular codes provide full transmit diversity and increase the capacity of wireless channels [**7, 29**].

3. Important Parameters: Maximum Rate and the Road to Minimum Delay

As with any combinatorial structure, there are natural optimization questions to ask concerning orthogonal designs. We discussed above the question of how many variables can be included in an $n \times n$ real orthogonal design. This question is generalized as follows: What is the *maximum rate* (ratio of number of variables to number of rows) achievable by a GCOD with n columns? A complementary question, known as the "fundamental problem for generalized complex orthogonal designs [**29**]," is as follows: What is the *minimum decoding delay* achievable by a maximum rate generalized complex orthogonal design with n columns?

Taken together, we see that the overarching goal for any number of columns is to arrange as many variables as possible into as few rows as possible. In other words, for any number of transmit antennas, we would like to send as many distinct signals as possible in the shortest possible amount of time. Of course, this optimization is subject to the orthogonality constraint on the columns.

Liang answered the first main research question by proving that GCODs with $2m-1$ or $2m$ columns have the same maximum rate of $\frac{m+1}{2m}$, where m is any natural number [**21**]. Furthermore, Liang provided an algorithm for constructing maximum rate GCODs for any number of columns [**21**]. Several other authors also worked towards determining the maximum rate and developing algorithms to produce high-rate GCODs [**28, 31, 32**].

Liang's result was hot off the press when I became interested in orthogonal designs and their associated codes, thus it was clear that I should tackle the question on minimum decoding delay. At that time, the only cases for which the minimum delay was known were the trivial cases of 2, 3, and 4 columns, the cases of 5 and 6 columns addressed by Liang [**21**], and the cases of 7 and 8 columns addressed by Kan and Shen [**19**]. The arguments for these cases were specialized to the number of columns, and thus could not be elegantly generalized.

My initial strategy was to focus on the combinatorial structure of maximum rate codes, hoping to exploit any found patterns and/or substructures. As I can tend to be old-fashioned, I began with a by-hand analysis of all known maximum rate GCODs. I looked for hidden patterns by printing out examples, literally cutting and taping rows and columns in different orders, and using colored highlighters to track various variables. Indeed, this "arts and crafts project" led to the identification of beautiful patterns and substructures within the maximum rate GCODs. The next challenge, of course, was to determine which patterns and substructures were significant and what they might imply about the minimum delay. This challenge was compounded by the size of the matrices under consideration: Even a "small" example has 8 columns and 112 rows, and every possible permutation of rows and columns ought to have been considered. It was clearly time to turn to computational power, a perfect project for undergraduate research students.

As luck would have it, just around this time, two Olin sophomores Nathan Karst and Jon Pollack came into my office looking for some advice on choosing a topic for their course project in my Discrete Mathematics class. (For the record, I believe their project proposal was due the next day.) These students were talented programmers, and I suggested that they implement Liang's new algorithm for maximum rate GCODs and create software to manipulate the generated designs. I showed them what I had been doing by hand: The avid programmers were horrified that I would do such work manually, and they left my office motivated to create software to automate all of my pattern-searching actions. Soon, they were hooked on the excitement of working on unsolved problems, and they became my research partners for the rest of their undergraduate careers (and beyond, in the case of Karst).

The new (and constantly evolving) software allowed us to generate examples larger than those previously available, which allowed us to test our hypotheses on larger sets of data. Even more importantly, the software allowed us to quickly reconfigure particular examples to identify additional patterns and substructures. We ultimately focused on the zero patterns of the rows in the maximum rate designs. A key observation was that every possible pattern of $m-1$ zeros in a row of length $2m$ seemed to appear in at least one row of any maximum rate GCOD with $2m$ columns. Linear algebraic and combinatorial arguments were employed to prove that this observation holds. This then led naturally to the result that a lower bound on the decoding delay for a maximum rate GCOD with $2m$ columns is $\binom{2m}{m-1}$ [**4**]. We later extended this result to hold for the case of $2m-1$ columns [**4**].

Naturally, we then needed to determine if our bound was tight. For the cases of $2m-1$ columns and $2m$ columns with m even, an algorithm already existed to generate maximum rate codes that happened to achieve the lower bound on delay [**28**]. For the case of $2m$ columns with m odd, however, no such algorithms existed. In fact, all known algorithms for maximum rate GCODs generated examples that achieved twice our lower bound on delay when the number of columns was $2m$ for m odd [**21, 28**]. Also, Liang had proven that for the case of 6 columns, the minimum decoding delay was 30, or twice our lower bound [**21**].

This case of $2m$ columns with m odd was elusive for quite a while, although we did have an early success in proving that if such a GCOD does not meet the lower bound, then it can at best meet twice the lower bound. Since algorithms did exist that happened to generate maximum rate codes that meet twice the lower bound in this case [**21, 28**], we were left to prove that GCODs in this case cannot achieve the lower bound.

After months of analysis, it became clear that we needed some new tools to attack this final case of $2m$ columns for m odd. Several new undergraduate research students had rotated through the group during this time, two of whom, Alex Dorsk and Andy Kalcic, helped developed a theory connecting GCODs to signed graphs, which are graphs whose edges are assigned weights of -1 or 1. Unfortunately, the restatement in terms of signed graphs turned out to be an unsolved problem studied (and known to be difficult) by graph theorists. However, this new paradigm allowed for new methods of attack, and it pleased me that my students were learning to initiate creative approaches to the problem.

In the end, the beast was tackled using a combination of combinatorial analysis, the signed graph paradigm, and a linear algebraic technique that was developed primarily by another undergraduate, Mathav Kishore Murugan, who was visiting me as a summer research student from the Indian Institute of Technology in Kharagpur, India. This was truly a collaborative effort, involving contributions from several undergraduates with complementary strengths and interests. The final result was as expected: For a maximum rate GCOD with $2m$ columns for m odd, the best achievable decoding delay is $2\binom{2m}{m-1}$, or twice the lower bound.

Related results came as corollaries to our main work on minimum decoding delay. For example, we showed that a maximum rate GCOD with $2m-1$ or $2m$ columns must utilize at least $\frac{1}{2}\binom{2m}{m}$ variables, with this bound being tight for $2m-1$ columns and $2m$ columns for m even. Similar to the result on minimum decoding delay, for the case of $2m$ columns with m odd, the minimum required number of variables is twice this bound, or $\binom{2m}{m}$ [**4**].

4. Additional Research Topics

4.1. Rate 1/2 Complex Orthogonal Designs. The minimum delay of $\binom{2m}{m-1}$ for a maximum rate GCOD with $2m-1$ or $2m$ columns grows quickly with respect to the number of columns. This suggests that non-rate-optimal GCODs may be preferable in practice (or more interesting in theory), as the decoding delay is prohibitively large in maximum rate GCODs with large numbers of columns. It is known that the maximum rate of GCODs with $2m-1$ or $2m$ columns is $\frac{m+1}{2m}$ [**21**], which approaches 1/2 as the number of columns increases. Hence there are two natural questions to ask: What is the minimum decoding delay for rate 1/2 codes? Would the sacrifice in rate be worth the trade-off for a smaller decoding delay? While these two questions are motivated by the application of designs to coding systems, the first question in particular reveals some interesting mathematics.

Several undergraduates have been involved in the effort to determine the minimum decoding delay of rate 1/2 GCODs, including Matthew Crawford, Caitlin Greeley, Nathan Karst, Mathav Kishore Murugan, and Bryce Lee. Our current work indicates that the minimum delay for a rate 1/2 GCOD with $2m-1$ or $2m$ columns is 2^{m-1} or 2^m, depending on the number of columns modulo 8. Although this delay is significantly smaller than the delay for maximum rate codes, it still grows exponentially.

It remains to examine the performance of these rate 1/2 codes, and it remains to answer whether we can slightly decrease the rate below 1/2 in order to gain a significant savings on decoding delay.

4.2. Conjugation Patterns and Transceiver Signal Linearization. As is often the case, a question that seems to have only theoretical importance turns out to have practical significance as well. During our work determining the minimum decoding delay for maximum rate generalized complex orthogonal designs, we began to notice a connection between whether a given design met the bound on decoding delay and whether it was possible to arrange the matrix using equivalence operations so that every nonzero entry in a given row was either conjugated or non-conjugated. We studied various patterns of conjugation mostly out of mathematical curiosity, however this property is also useful in the application of these designs as COSTBCs.

Certain COSTBCs enjoy a property known as *transceiver signal linearization*, which can facilitate decoding: This linearization allows the code to be backward compatible with existing signal processing techniques and standards, and it allows for the design of low complexity channel equalizers and interference suppressing filters [**27**]. It has been shown that a COSTBC can achieve transceiver signal linearization if each row in the code has either all conjugated entries or all non-conjugated entries [**27**], which is precisely the property we analyzed while looking for mathematical patterns in the underlying designs.

We have been able to show that if a maximum rate COSTBC has an odd number of columns $2m-1$ and achieves the minimum delay of $\binom{2m}{m-1}$ or has an even number of columns $2m$ and achieves twice the minimum delay $2\binom{2m}{m-1}$, then transceiver signal linearization is achievable. If a maximum rate COSTBC has an even number of columns $2m$ and achieves the minimum delay, then transceiver signal linearization is not possible.

This work highlights the fact that trade-offs are omnipresent when designing a communication system. For example, there are maximum rate COSTBCs with $2m \equiv 0 \pmod 4$ columns that achieve transceiver signal linearization, and there are such codes that achieve the lower bound on decoding delay, but no such code can achieve both properties simultaneously.

Our current work examines which rate 1/2 COSTBCs can achieve transceiver signal linearization, and again the work indicates that the answer depends on the number of columns modulo 8.

5. Conclusions

This paper has provided a gentle introduction to orthogonal designs and their application as complex orthogonal space-time block codes. We have reviewed the major results of both theoretical and practical importance regarding these designs/codes, presenting the maximum rate as determined by Liang and providing a narrative of the determination of the minimum decoding delay by this author working with an army of undergraduates. We have also briefly described certain related research topics, including some open problems.

Recent breakthroughs in antenna technology [**15, 30**] and a growing interest in distributed systems contribute to a growing interest in determining the optimal trade-off between rate and delay for COSTBCs with arbitrary numbers of antennas. However, we must note that it is only for two antennas that we can simultaneously achieve full rate, square size (meaning lowest possible delay), and transceiver linearization. As the number of antennas increases, there is a sacrifice in some or all of these parameters. The analysis of such trade-offs is at the core of the challenge of designing wireless communication systems.

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