

# MATH312: Applied Mathematical Modelling III

## Solutions to the 2007 final exam test

### Question 1:

(a) (i) Required to prove:  $\underline{u} \times (\underline{v}\underline{w}) = (\underline{u} \times \underline{v})\underline{w}$ . Thus,

$$\begin{aligned} \text{LHS} &= (u_i \underline{e}_i) \times [(v_j \underline{e}_j)(w_k \underline{e}_k)], & \text{RHS} &= [(u_i \underline{e}_i) \times (v_j \underline{e}_j)](w_k \underline{e}_k), \\ &= (u_i \underline{e}_i) \times [v_j w_k \underline{e}_j \underline{e}_k], & &= [u_i v_j \varepsilon_{ijm} \underline{e}_m] w_k \underline{e}_k, \\ &= u_i v_j w_k (\underline{e}_i \times \underline{e}_j) \underline{e}_k, & &= u_i v_j w_k \varepsilon_{ijm} \underline{e}_m \underline{e}_k = \text{LHS}. \\ &= u_i v_j w_k \varepsilon_{ijm} \underline{e}_m \underline{e}_k. \end{aligned}$$

Therefore,  $\underline{u} \times (\underline{v}\underline{w}) = (\underline{u} \times \underline{v})\underline{w}$ .

(ii) Required to prove:  $\underline{\nabla} \times (f\underline{u}) = (\underline{\nabla}f) \times \underline{u} + f(\underline{\nabla} \times \underline{u})$ . Thus,

$$\begin{aligned} \text{LHS} &= \left( \underline{e}_m \frac{\partial}{\partial x_m} \right) \times (f u_i \underline{e}_i), & \text{RHS} &= (\underline{e}_m f_{,m}) \times (u_i \underline{e}_i) + f \left[ \left( \underline{e}_m \frac{\partial}{\partial x_m} \right) \times (u_i \underline{e}_i) \right], \\ &= \frac{\partial}{\partial x_m} (f u_i) (\underline{e}_m \times \underline{e}_i), & &= f_{,m} u_i (\underline{e}_m \times \underline{e}_i) + f u_{i,m} (\underline{e}_m \times \underline{e}_i), \\ &= (f_{,m} u_i + f u_{i,m}) \varepsilon_{min} \underline{e}_n. & &= (f_{,m} u_i + f u_{i,m}) \varepsilon_{min} \underline{e}_n = \text{LHS}. \end{aligned}$$

Therefore,  $\underline{\nabla} \times (f\underline{u}) = (\underline{\nabla}f) \times \underline{u} + f(\underline{\nabla} \times \underline{u})$ .

(iii) Required to prove:  $\underline{u}(\underline{v} \cdot \underline{w} \times \underline{r}) - \underline{v}(\underline{u} \cdot \underline{w} \times \underline{r}) = -(\underline{u} \times \underline{v}) \times (\underline{w} \times \underline{r})$ . Thus,

$$\begin{aligned} \text{LHS} &= (u_i \underline{e}_i)[(v_j \underline{e}_j) \cdot (w_k \underline{e}_k) \times (r_m \underline{e}_m)] - (v_j \underline{e}_j)[(u_i \underline{e}_i) \cdot (w_k \underline{e}_k) \times (r_m \underline{e}_m)], \\ &= (u_i \underline{e}_i)[(v_j \underline{e}_j) \cdot (w_k r_m \varepsilon_{kmo} \underline{e}_o)] - (v_j \underline{e}_j)[(u_i \underline{e}_i) \cdot (w_k r_m \varepsilon_{kmo} \underline{e}_o)], \\ &= (u_i \underline{e}_i)[v_j w_k r_m \varepsilon_{kmo} \delta_{jo}] - (v_j \underline{e}_j)[u_i w_k r_m \varepsilon_{kmo} \delta_{io}], \\ &= u_i v_j w_k r_m \varepsilon_{kmj} \underline{e}_i - u_j v_j w_k r_m \varepsilon_{kmi} \underline{e}_j. \end{aligned}$$

$$\begin{aligned} \text{RHS} &= -[(u_i \underline{e}_i) \times (v_j \underline{e}_j)] \times [(w_k \underline{e}_k) \times (r_m \underline{e}_m)], \\ &= -[u_i v_j \varepsilon_{ijo} \underline{e}_o] \times [w_k r_m \varepsilon_{kmp} \underline{e}_p], \\ &= -u_i v_j w_k r_m \varepsilon_{ijo} \varepsilon_{kmp} \varepsilon_{opq} \underline{e}_q, \\ &= -u_i v_j w_k r_m \varepsilon_{kmp} [\varepsilon_{oij} \varepsilon_{opq}] \underline{e}_q, \\ &= -u_i v_j w_k r_m \varepsilon_{kmp} [\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}] \underline{e}_q, \\ &= -u_i v_j w_k r_m \varepsilon_{kmi} \underline{e}_j + u_i v_j w_k r_m \varepsilon_{kmj} \underline{e}_i = \text{LHS}. \end{aligned}$$

Therefore,  $\underline{u}(\underline{v} \cdot \underline{w} \times \underline{r}) - \underline{v}(\underline{u} \cdot \underline{w} \times \underline{r}) = -(\underline{u} \times \underline{v}) \times (\underline{w} \times \underline{r})$ .

(b) (i) We know  $\alpha_j^i = \cos(\angle_{x^i, X^j})$ , so that

$$\begin{aligned}\alpha_1^1 &= \cos(\angle_{x^1, X^1}) = \cos 90^\circ = 0, \\ \alpha_2^1 &= \cos(\angle_{x^1, X^2}) = \cos 45^\circ = 1/\sqrt{2}, \\ \alpha_3^1 &= \cos(\angle_{x^1, X^3}) = \cos 135^\circ = -1/\sqrt{2}, \\ \\ \alpha_1^2 &= \cos(\angle_{x^2, X^1}) = \cos 45^\circ = 1/\sqrt{2}, \\ \alpha_2^2 &= \cos(\angle_{x^2, X^2}) = \cos 60^\circ = 1/2, \\ \alpha_3^2 &= \cos(\angle_{x^2, X^3}) = \cos 60^\circ = 1/2, \\ \\ \alpha_1^3 &= \cos(\angle_{x^3, X^1}) = \cos 45^\circ = 1/\sqrt{2}, \\ \alpha_2^3 &= \cos(\angle_{x^3, X^2}) = \cos 120^\circ = -1/2, \\ \alpha_3^3 &= \cos(\angle_{x^3, X^3}) = \cos 120^\circ = -1/2,\end{aligned}$$

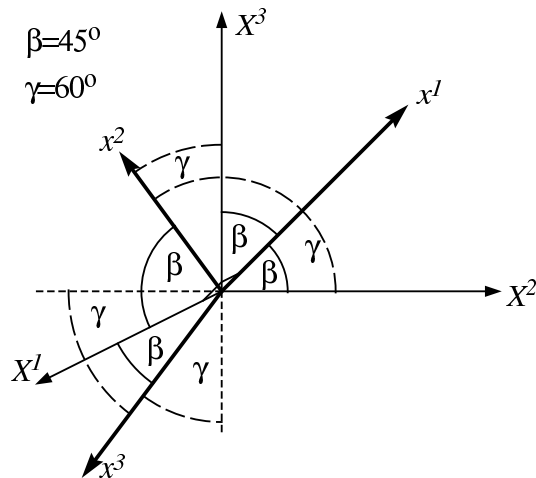
Therefore, we find

$$\alpha_j^i = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

(ii) We know  $x^i = \alpha_j^i X^j$ , where  $\alpha_j^i$  is given above and  $X = (0, 1, -1)$ . Thus,

$$\begin{aligned}x &= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2} & 0 & 0 \end{pmatrix}^T.\end{aligned}$$

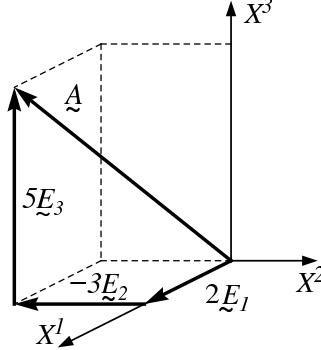
(iii)



**Question 2:**

$$\underline{A} = 2\underline{E}_1 - 3\underline{E}_2 + 5\underline{E}_3.$$

(a)



(b)  $\underline{A}$  starts at the origin, then moves  $2|\underline{E}_1|$  units in the direction of  $\underline{E}_1$ ,  $X^1$ , then  $-3|\underline{E}_2|$  units in the direction of  $\underline{E}_2$ ,  $X^2$ , and finally  $5|\underline{E}_3|$  units in the direction of  $\underline{E}_3$ ,  $X^3$ . The final point is  $(2|\underline{E}_1|, -3|\underline{E}_2|, 5|\underline{E}_3|)$ .

(c) The contravariant components are

$$A^1 = 2, \quad A^2 = -3, \quad A^3 = 5.$$

(d)  $\underline{A} = 2\underline{E}_1 - 3\underline{E}_2 + 5\underline{E}_3 = 2|\underline{E}_1|\frac{\underline{E}_1}{|\underline{E}_1|} - 3|\underline{E}_2|\frac{\underline{E}_2}{|\underline{E}_2|} + 5|\underline{E}_3|\frac{\underline{E}_3}{|\underline{E}_3|}$ , Therefore,

$$A^{(1)} = 2|\underline{E}_1|, \quad A^{(2)} = -3|\underline{E}_2|, \quad A^{(3)} = 5|\underline{E}_3|.$$

Further, as  $\frac{A_i}{A^i} = |\underline{E}_i|^2$ , no summation over  $i$ , then

$$A_i = A^i|\underline{E}_i|^2 \quad \Rightarrow \quad A_1 = 2|\underline{E}_1|^2, \quad A_2 = -3|\underline{E}_2|^2, \quad A_3 = 5|\underline{E}_3|^2,$$

so that as

$$\underline{A} = A_1\underline{E}^1 + A_2\underline{E}^2 + A_3\underline{E}^3 = A_1|\underline{E}^1|\frac{\underline{E}^1}{|\underline{E}^1|} + A_2|\underline{E}^2|\frac{\underline{E}^2}{|\underline{E}^2|} + A_3|\underline{E}^3|\frac{\underline{E}^3}{|\underline{E}^3|},$$

then

$$A_{(1)} = 2|\underline{E}_1|^2|\underline{E}^1|, \quad A_{(2)} = -3|\underline{E}_2|^2|\underline{E}^2|, \quad A_{(3)} = 5|\underline{E}_3|^2|\underline{E}^3|.$$

(e) If

$$x = \rho^2 \cos \vartheta, \quad y = \rho^2 \sin \vartheta, \quad z = \zeta,$$

then the position vector  $\underline{r}$  is given by

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = \rho^2 \cos \vartheta \underline{i} + \rho^2 \sin \vartheta \underline{j} + \zeta \underline{k}.$$

Thus, as  $\underline{E}_i = \frac{\partial \underline{r}}{\partial X^i}$  and  $(X^1, X^2, X^3) = (\rho, \vartheta, \zeta)$ , then

$$\begin{aligned}\underline{E}_1 &= \frac{\partial \underline{r}}{\partial X^1} = 2\rho \cos \vartheta \underline{i} + 2\rho \sin \vartheta \underline{j} = (2\rho \cos \vartheta, 2\rho \sin \vartheta, 0), \\ \underline{E}_2 &= \frac{\partial \underline{r}}{\partial X^2} = -\rho^2 \sin \vartheta \underline{i} + \rho^2 \cos \vartheta \underline{j} = (-\rho^2 \sin \vartheta, \rho^2 \cos \vartheta, 0), \\ \underline{E}_3 &= \frac{\partial \underline{r}}{\partial X^3} = \underline{k} = (0, 0, 1).\end{aligned}$$

(f) Previously, we found that the covariant components of  $\underline{A}$  are given by

$$A_1 = 2|\underline{E}_1|^2, \quad A_2 = -3|\underline{E}_2|^2, \quad A_3 = 5|\underline{E}_3|^2,$$

so that

$$\begin{aligned}A_1 &= 2 \left( \sqrt{4\rho^2 \cos^2 \vartheta + 4\rho^2 \sin^2 \vartheta + 0^2} \right)^2 = 8\rho^2, \\ A_2 &= -3 \left( \sqrt{\rho^4 \sin^2 \vartheta + \rho^4 \cos^2 \vartheta + 0^2} \right)^2 = -3\rho^4, \\ A_3 &= 5 \left( \sqrt{0^2 + 0^2 + 1^2} \right)^2 = 5.\end{aligned}$$

(g) We know  $\underline{E}^1 = \frac{1}{V}(\underline{E}_2 \times \underline{E}_3)$ , where  $V = \underline{E}_1 \cdot (\underline{E}_2 \times \underline{E}_3)$ . Thus,

$$V = \begin{vmatrix} 2\rho \cos \vartheta & 2\rho \sin \vartheta & 0 \\ -\rho^2 \sin \vartheta & \rho^2 \cos \vartheta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\rho^3(\cos^2 \vartheta + \sin^2 \vartheta) = 2\rho^3.$$

Therefore,

$$\begin{aligned}
\tilde{E}^1 &= \frac{1}{V}(\tilde{E}_2 \times \tilde{E}_3) = \frac{1}{2\rho^3} \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ -\rho^2 \sin \vartheta & \rho^2 \cos \vartheta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2\rho^3}(\rho^2 \cos \vartheta \tilde{i} + \rho^2 \sin \vartheta \tilde{j}), \\
&= \frac{\cos \vartheta}{2\rho} \tilde{i} + \frac{\sin \vartheta}{2\rho} \tilde{j} = \left( \frac{\cos \vartheta}{2\rho}, \frac{\sin \vartheta}{2\rho}, 0 \right), \\
\tilde{E}^2 &= \frac{1}{V}(\tilde{E}_3 \times \tilde{E}_1) = \frac{1}{2\rho^3} \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 0 & 0 & 1 \\ 2\rho \cos \vartheta & 2\rho \sin \vartheta & 0 \end{vmatrix} = \frac{1}{2\rho^3}(-2\rho \sin \vartheta \tilde{i} + 2\rho \cos \vartheta \tilde{j}), \\
&= -\frac{\sin \vartheta}{\rho^2} \tilde{i} + \frac{\cos \vartheta}{\rho^2} \tilde{j} = \left( -\frac{\sin \vartheta}{\rho^2}, \frac{\cos \vartheta}{\rho^2}, 0 \right), \\
\tilde{E}^3 &= \frac{1}{V}(\tilde{E}_2 \times \tilde{E}_1) = \frac{1}{2\rho^3} \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 2\rho \cos \vartheta & 2\rho \sin \vartheta & 0 \\ -\rho^2 \sin \vartheta & \rho^2 \cos \vartheta & 0 \end{vmatrix} = \frac{1}{2\rho^3} 2\rho^3 (\cos^2 \vartheta + \sin^2 \vartheta) \tilde{k}, \\
&= \tilde{k} = (0, 0, 1).
\end{aligned}$$

(h) We know

$$\begin{aligned}
A^{(1)} &= 2|\tilde{E}_1|, & A^{(2)} &= -3|\tilde{E}_2|, & A^{(3)} &= 5|\tilde{E}_3|, \\
A_{(1)} &= 2|\tilde{E}_1|^2|\tilde{E}^1|, & A_{(2)} &= -3|\tilde{E}_2|^2|\tilde{E}^2|, & A_{(3)} &= 5|\tilde{E}_3|^2|\tilde{E}^3|.
\end{aligned}$$

and

$$\begin{aligned}
\tilde{E}_1 &= (2\rho \cos \vartheta, 2\rho \sin \vartheta, 0), & \tilde{E}_2 &= (-\rho^2 \sin \vartheta, \rho^2 \cos \vartheta, 0), & \tilde{E}_3 &= (0, 0, 1), \\
\tilde{E}^1 &= \left( \frac{\cos \vartheta}{2\rho}, \frac{\sin \vartheta}{2\rho}, 0 \right), & \tilde{E}^2 &= \left( -\frac{\sin \vartheta}{\rho^2}, \frac{\cos \vartheta}{\rho^2}, 0 \right), & \tilde{E}^3 &= (0, 0, 1).
\end{aligned}$$

Thus,

$$\begin{aligned}
A^{(1)} &= 2\sqrt{4\rho^2 \cos^2 \vartheta + 4\rho^2 \sin^2 \vartheta} & A^{(2)} &= -3\sqrt{\rho^4 \sin^2 \vartheta + \rho^4 \cos^2 \vartheta} & A^{(3)} &= 5\sqrt{1} \\
&= 4\rho, & &= -3\rho^2, & &= 5,
\end{aligned}$$

$$\begin{aligned}
A_{(1)} &= 2(4\rho^2) \sqrt{\frac{\cos^2 \vartheta}{4\rho^2} + \frac{\sin^2 \vartheta}{4\rho^2}} & A_{(2)} &= -3(\rho^4) \sqrt{\frac{\sin^2 \vartheta}{\rho^4} + \frac{\cos^2 \vartheta}{\rho^4}} & A_{(3)} &= 5(1)\sqrt{1} \\
&= 2(4\rho^2) \left( \frac{1}{2\rho} \right) & &= -3(\rho^4) \left( \frac{1}{\rho^2} \right) & &= 1. \\
&= 4\rho, & &= -3\rho^2, & &
\end{aligned}$$

Therefore,

$$A^{(1)} = A_{(1)} = 4\rho, \quad A^{(2)} = A_{(2)} = -3\rho^2, \quad A^{(3)} = A_{(3)} = 5.$$

(i) The physical components found in (h) above represent how many unit lengths the vector  $\underline{A}$  moves in the respective directions.

(j) For a reciprocal basis, we want  $\underline{E}_i \cdot \underline{E}^j = \delta_i^j$  ( $= \underline{E}^j \cdot \underline{E}_i$ ). Thus,

$$\underline{E}_1 \cdot \underline{E}^1 = (2\rho \cos \vartheta, 2\rho \sin \vartheta, 0) \cdot \left( \frac{\cos \vartheta}{2\rho}, \frac{\sin \vartheta}{2\rho}, 0 \right) = \frac{2\rho \cos^2 \vartheta}{2\rho} + \frac{2\rho \sin^2 \vartheta}{2\rho} + 0 = 1 = \delta_1^1 \quad \checkmark$$

$$\underline{E}_1 \cdot \underline{E}^2 = (2\rho \cos \vartheta, 2\rho \sin \vartheta, 0) \cdot \left( -\frac{\sin \vartheta}{\rho^2}, \frac{\cos \vartheta}{\rho^2}, 0 \right) = -\frac{2\rho \sin \vartheta \cos \vartheta}{\rho^2} + \frac{2\rho \sin \vartheta \cos \vartheta}{\rho^2} = 0 = \delta_1^2 \quad \checkmark$$

$$\underline{E}_2 \cdot \underline{E}^1 = (-\rho^2 \sin \vartheta, \rho^2 \cos \vartheta, 0) \cdot \left( \frac{\cos \vartheta}{2\rho}, \frac{\sin \vartheta}{2\rho}, 0 \right) = -\frac{\rho^2 \sin \vartheta \cos \vartheta}{2\rho} + \frac{\rho^2 \sin \vartheta \cos \vartheta}{2\rho} = 0 = \delta_2^1 \quad \checkmark$$

$$\underline{E}_2 \cdot \underline{E}^2 = (-\rho^2 \sin \vartheta, \rho^2 \cos \vartheta, 0) \cdot \left( -\frac{\sin \vartheta}{\rho^2}, \frac{\cos \vartheta}{\rho^2}, 0 \right) = \frac{\rho^2 \sin^2 \vartheta}{\rho^2} + \frac{\rho^2 \cos^2 \vartheta}{\rho^2} = 1 = \delta_2^2 \quad \checkmark$$

$$\underline{E}_3 \cdot \underline{E}^3 = (0, 0, 1) \cdot (0, 0, 1) = 1 = \delta_3^3 \quad \checkmark$$

where  $\underline{E}_3 \cdot \underline{E}_1 = \underline{E}_1 \cdot \underline{E}^3 = \underline{E}_3 \cdot \underline{E}^2 = \underline{E}_2 \cdot \underline{E}^3 = 0$  are obvious.

Therefore, the tangential and gradient basis vectors form a set of reciprocal basis.

(k) Recall that

$$\begin{aligned} \underline{E}_1 &= (2\rho \cos \vartheta, 2\rho \sin \vartheta, 0), & \underline{E}_2 &= (-\rho^2 \sin \vartheta, \rho^2 \cos \vartheta, 0), & \underline{E}_3 &= (0, 0, 1), \\ \underline{E}^1 &= \left( \frac{\cos \vartheta}{2\rho}, \frac{\sin \vartheta}{2\rho}, 0 \right), & \underline{E}^2 &= \left( -\frac{\sin \vartheta}{\rho^2}, \frac{\cos \vartheta}{\rho^2}, 0 \right), & \underline{E}^3 &= (0, 0, 1). \end{aligned}$$

Thus, if  $\underline{E}_i = k_i \underline{E}^i$ , then by observation

$$k_1 = 4\rho^2, \quad k_2 = \rho^4, \quad k_3 = 1.$$

Further, upon examining the metric tensor, namely

$$g_{ij} = \underline{E}_i \cdot \underline{E}_j = \begin{pmatrix} 4\rho^2 & 0 & 0 \\ 0 & \rho^4 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then we see

$$k_1 = g_{11}, \quad k_2 = g_{22}, \quad k_3 = g_{33},$$

which should not be surprising, as  $\underline{E}_i = g_{ij} \underline{E}^j$ .

### Question 3:

(a) If  $x^1 = 2X^1 + X^2$ ,  $x^2 = X^1 + X^2$  and  $x^3 = X^3$ , then the deformation matrix is given by

$$\mathbf{F} = \left( \frac{\partial x^i}{\partial X^j} \right) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) If  $\mathbf{U}$  and  $\mathbf{V}$  are symmetric, then  $\mathbf{U}^T = \mathbf{U}$  and  $\mathbf{V}^T = \mathbf{V}$ . Further, if  $\mathbf{R}$  is a rotation matrix, then  $\mathbf{R}^T \mathbf{R} = \mathbf{I}$ .

(i) We know  $\mathbf{F} = \mathbf{R}\mathbf{U}$ , so that

$$\begin{aligned} \mathbf{F}^T \mathbf{F} &= (\mathbf{R}\mathbf{U})^T (\mathbf{R}\mathbf{U}), \\ &= \mathbf{U}^T \mathbf{R}^T \mathbf{R} \mathbf{U}, \\ &= \mathbf{U}^T \mathbf{U}, & (\mathbf{R}^T \mathbf{R} = \mathbf{I}) \\ &= \mathbf{U}^2, & (\mathbf{U}^T = \mathbf{U}) \end{aligned}$$

i.e., 
$$\mathbf{U} = \sqrt{\mathbf{F}^T \mathbf{F}}.$$

(ii) Similarly, we know  $\mathbf{F} = \mathbf{V}\mathbf{R}$ , so that

$$\begin{aligned} \mathbf{F}\mathbf{F}^T &= (\mathbf{V}\mathbf{R})(\mathbf{V}\mathbf{R})^T, \\ &= \mathbf{V}\mathbf{R}\mathbf{R}^T \mathbf{V}^T, \\ &= \mathbf{V}\mathbf{V}^T, & (\mathbf{R}\mathbf{R}^T = \mathbf{I}) \\ &= \mathbf{V}^2, & (\mathbf{V}^T = \mathbf{V}) \end{aligned}$$

i.e., 
$$\mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T}.$$

(c) If  $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ , then

$$\mathbf{C} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, the eigenvalues are found by solving  $|\mathbf{C} - \lambda \mathbf{I}| = 0$ , i.e.,

$$\begin{vmatrix} 5 - \lambda & 3 & 0 \\ 3 & 2 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = 0,$$

$$(1 - \lambda)[(5 - \lambda)(2 - \lambda) - 9] = 0,$$

$$(1 - \lambda)(\lambda^2 - 7\lambda + 1) = 0,$$

i.e., 
$$\lambda = 1, \frac{7}{2} + \frac{3\sqrt{5}}{2}, \frac{7}{2} - \frac{3\sqrt{5}}{2}.$$

To find the corresponding eigenvectors, consider  $(\mathbf{C} - \lambda \mathbf{I})\underline{x} = \underline{0}$  for each eigenvalue. Thus:

1.  $\lambda_1 = \frac{7}{2} + \frac{3\sqrt{5}}{2}$  :

$$\begin{aligned} & \begin{pmatrix} 5 - \frac{7}{2} - \frac{3\sqrt{5}}{2} & 3 & 0 \\ 3 & 2 - \frac{7}{2} - \frac{3\sqrt{5}}{2} & 0 \\ 0 & 0 & 1 - \frac{7}{2} - \frac{3\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ \Rightarrow & \begin{pmatrix} \frac{3}{2} - \frac{3\sqrt{5}}{2} & 3 & 0 \\ 3 & -\frac{3}{2} - \frac{3\sqrt{5}}{2} & 0 \\ 0 & 0 & -\frac{5}{2} - \frac{3\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

which implies that  $x_3 = 0$ , and

$$\begin{aligned} x_1 &= -\frac{3}{\frac{3}{2} - \frac{3\sqrt{5}}{2}}x_2 = \frac{\frac{3}{2} + \frac{3\sqrt{5}}{2}}{3}x_2, \\ \Rightarrow & \left(\frac{9}{4} - \frac{45}{4} + 9\right)x_2 = 0, \\ \text{i.e.,} & \quad \quad \quad 0 \times x_2 = 0, \end{aligned}$$

so there is a non-trivial solution. Thus, let  $x_1 = t$  so that

$$x_2 = \left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right)t.$$

Therefore,  $\underline{x}_1 = \left(1, -\frac{1}{2} + \frac{\sqrt{5}}{2}, 0\right)^T t$ .

2.  $\lambda_2 = \frac{7}{2} - \frac{3\sqrt{5}}{2}$  :

$$\begin{aligned} & \begin{pmatrix} 5 - \frac{7}{2} + \frac{3\sqrt{5}}{2} & 3 & 0 \\ 3 & 2 - \frac{7}{2} + \frac{3\sqrt{5}}{2} & 0 \\ 0 & 0 & 1 - \frac{7}{2} + \frac{3\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ \Rightarrow & \begin{pmatrix} \frac{3}{2} + \frac{3\sqrt{5}}{2} & 3 & 0 \\ 3 & -\frac{3}{2} + \frac{3\sqrt{5}}{2} & 0 \\ 0 & 0 & -\frac{5}{2} + \frac{3\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

which implies that  $x_3 = 0$ , and

$$\begin{aligned} x_1 &= -\frac{3}{\frac{3}{2} + \frac{3\sqrt{5}}{2}}x_2 = \frac{\frac{3}{2} - \frac{3\sqrt{5}}{2}}{3}x_2, \\ \Rightarrow & \left(\frac{9}{4} - \frac{45}{4} + 9\right)x_2 = 0, \\ \text{i.e.,} & \quad \quad \quad 0 \times x_2 = 0, \end{aligned}$$

so there is a non-trivial solution. Thus, let  $x_1 = t$  so that

$$x_2 = \left( -\frac{1}{2} - \frac{\sqrt{5}}{2} \right) t.$$

Therefore,  $\underline{x}_2 = \left( 1, -\frac{1}{2} - \frac{\sqrt{5}}{2}, 0 \right)^T t$ .

3.  $\lambda_3 = 1$  :

$$\begin{aligned} & \begin{pmatrix} 5-1 & 3 & 0 \\ 3 & 2-1 & 0 \\ 0 & 0 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ \Rightarrow & \begin{pmatrix} 4 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

which implies that  $x_3 = t$ , and

$$\begin{aligned} & x_1 = -\frac{3}{4}x_2 = \frac{1}{3}x_2, \\ \Rightarrow & 13x_2 = 0, \\ \text{i.e.,} & x_2 = 0, \end{aligned}$$

so there is only a non-trivial solution for  $x_2$  and  $x_1$ . Therefore,  $\underline{x}_3 = (0, 0, 1)^T t$ .

(d) We know

$$\mathbf{D} = \begin{pmatrix} \frac{7}{2} + \frac{3\sqrt{5}}{2} & 0 & 0 \\ 0 & \frac{7}{2} - \frac{3\sqrt{5}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & 0 \\ \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Considering  $\mathbf{P}\mathbf{P}$  gives

$$\mathbf{P}\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & 0 \\ \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & 0 \\ \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) Therefore, from 3(di),  $\mathbf{P}^{-1} = \mathbf{P}$ , i.e.,

$$\mathbf{P}^{-1} = \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & 0 \\ \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iii) We consider the matrix decomposition given in (1) because it allows us to calculate the square root of  $\mathbf{C}$ , which in turn will give us the right and left stretch tensors, and the rotation matrix of the deformation.

(e) To find  $\sqrt{\mathbf{C}}$ , we know that  $\sqrt{\mathbf{C}} = \mathbf{P}\sqrt{\mathbf{D}}\mathbf{P}^{-1}$ , so that

$$\begin{aligned}\sqrt{\mathbf{C}} &= \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & 0 \\ \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{7}{2} + \frac{3\sqrt{5}}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{7}{2} - \frac{3\sqrt{5}}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & 0 \\ \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ &= \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & 0 \\ \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} + \frac{\sqrt{5}}{2} & 0 & 0 \\ 0 & \frac{3}{2} - \frac{\sqrt{5}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & 0 \\ \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ &= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

Therefore, the right stretch tensor is

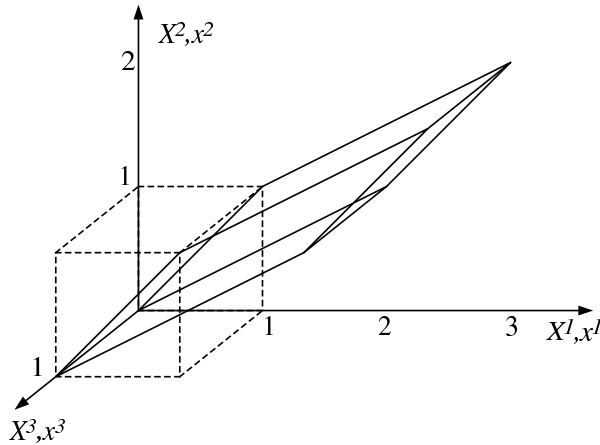
$$\mathbf{U} = \sqrt{\mathbf{C}} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (= \mathbf{F})$$

(f) To find the rotation matrix, you can either find  $\mathbf{U}^{-1}$  and then  $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$ , or notice from (e) that  $\mathbf{U} = \mathbf{F}$ , which implies from  $\mathbf{F} = \mathbf{R}\mathbf{U}$  that  $\mathbf{R} = \mathbf{I}$ .

Further, if  $\mathbf{R} = \mathbf{I}$ , then from  $\mathbf{F} = \mathbf{V}\mathbf{R}$ , we see that  $\mathbf{V} = \mathbf{F} = \mathbf{U}$ .

This says that the deformation is a pure deformation, with no rotation.

(g)



The resulting deformed body of material has been stretch solely in the  $X^1 - X^2$  plane, where the shape has sort of been “pulled” along the  $X^2 = 2X^1$  direction.

## Question 4:

We know

$$\begin{aligned}\sigma_{xx} &= k[xy - \nu(x^2 - y^2)], \\ \sigma_{yy} &= k[yx - \nu(y^2 - x^2)], \\ \sigma_{zz} &= kxy, \\ \sigma_{xy} &= \frac{k}{2}[4\nu xy - x^2 - y^2], \\ \sigma_{xz} &= \sigma_{yz} = 0.\end{aligned}$$

(a). The generalized Hooke's Law for elastic isotropic materials is

$$\epsilon_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij},$$

where  $i, j = x, y, z$ . Thus, as  $\sigma_{kk} = 3kxy$ , then

$$\begin{aligned}\epsilon_{xx} &= \frac{1 + \nu}{E} \{k[xy - \nu(x^2 - y^2)]\} - \frac{\nu}{E} (3kxy) \delta_{xx} = \frac{1 - 2\nu}{E} kxy - \frac{1 + \nu}{E} \nu(x^2 - y^2), \\ \epsilon_{yy} &= \frac{1 + \nu}{E} \{k[xy - \nu(y^2 - x^2)]\} - \frac{\nu}{E} (3kxy) \delta_{yy} = \frac{1 - 2\nu}{E} kxy - \frac{1 + \nu}{E} \nu(y^2 - x^2), \\ \epsilon_{zz} &= \frac{1 + \nu}{E} \{kxy\} - \frac{\nu}{E} (3kxy) \delta_{zz} = \frac{1 - 2\nu}{E} kxy, \\ \epsilon_{xy} &= \frac{1 + \nu}{E} \left\{ \frac{k}{2} [4\nu xy - x^2 - y^2] \right\} - \frac{\nu}{E} (3kxy) \delta_{xy} = \frac{k(1 + \nu)}{2E} [4\nu xy - x^2 - y^2], \\ \epsilon_{xz} &= \frac{1 + \nu}{E} \{0\} - \frac{\nu}{E} (3kxy) \delta_{xz} = 0, \\ \epsilon_{yz} &= \frac{1 + \nu}{E} \{0\} - \frac{\nu}{E} (3kxy) \delta_{yz} = 0.\end{aligned}$$

(b). Consider

$$\epsilon_{pks} (\epsilon_{sj,ik} - \epsilon_{si,jk}) = 0.$$

There are 3 free indices  $(p, i, j)$  so there should be  $3^3 = 27$  equations. However, we can eliminate cases  $i = j$  where the equations are automatically satisfied. By inspection, we see that symmetry exists between  $i$  and  $j$ , so we only need to consider the cases  $i = 1, j = 2; i = 1, j = 3; i = 2, j = 3$ . Thus, we have 9 equations (let  $p = 1, 2, 3$  for the above 3 cases).

$p = 1 : i = 1, j = 2$

$$\begin{aligned}\epsilon_{123} \left( \frac{\partial^2 \epsilon_{32}}{\partial X^1 \partial X^2} - \frac{\partial^2 \epsilon_{31}}{\partial X^2 \partial X^2} \right) + \epsilon_{132} \left( \frac{\partial^2 \epsilon_{22}}{\partial X^1 \partial X^3} - \frac{\partial^2 \epsilon_{21}}{\partial X^2 \partial X^3} \right) &= 0, \\ \Rightarrow \frac{\partial^2 \epsilon_{32}}{\partial X^1 \partial X^2} + \frac{\partial^2 \epsilon_{21}}{\partial X^2 \partial X^3} &= \frac{\partial^2 \epsilon_{31}}{\partial X^2} + \frac{\partial^2 \epsilon_{22}}{\partial X^1 \partial X^3}.\end{aligned}\tag{1}$$

$$p = 1 : i = 1, j = 3$$

$$\begin{aligned} & \varepsilon_{123} \left( \frac{\partial^2 \epsilon_{33}}{\partial X^1 \partial X^2} - \frac{\partial^2 \epsilon_{31}}{\partial X^3 \partial X^2} \right) + \varepsilon_{132} \left( \frac{\partial^2 \epsilon_{23}}{\partial X^1 \partial X^3} - \frac{\partial^2 \epsilon_{21}}{\partial X^3 \partial X^3} \right) = 0 \\ \Rightarrow & \frac{\partial^2 \epsilon_{33}}{\partial X^1 \partial X^2} + \frac{\partial^2 \epsilon_{21}}{\partial X_3^2} = \frac{\partial^2 \epsilon_{31}}{\partial X^2 \partial X^3} + \frac{\partial^2 \epsilon_{23}}{\partial X^1 \partial X^3}. \end{aligned} \quad (2)$$

$$p = 1 : i = 2, j = 3$$

$$\begin{aligned} & \varepsilon_{123} \left( \frac{\partial^2 \epsilon_{33}}{\partial X^2 \partial X^2} - \frac{\partial^2 \epsilon_{32}}{\partial X^3 \partial X^2} \right) + \varepsilon_{132} \left( \frac{\partial^2 \epsilon_{23}}{\partial X^2 \partial X^3} - \frac{\partial^2 \epsilon_{22}}{\partial X^3 \partial X^3} \right) = 0, \\ \Rightarrow & \frac{\partial^2 \epsilon_{33}}{\partial X_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial X_3^2} = \frac{\partial^2 \epsilon_{32}}{\partial X^2 \partial X^3} + \frac{\partial^2 \epsilon_{23}}{\partial X^2 \partial X^3} = 2 \frac{\partial^2 \epsilon_{23}}{\partial X^2 \partial X^3}. \end{aligned} \quad (3)$$

$$p = 2 : i = 1, j = 2$$

$$\begin{aligned} & \varepsilon_{213} \left( \frac{\partial^2 \epsilon_{32}}{\partial X^1 \partial X^1} - \frac{\partial^2 \epsilon_{31}}{\partial X^2 \partial X^1} \right) + \varepsilon_{231} \left( \frac{\partial^2 \epsilon_{12}}{\partial X^1 \partial X^3} - \frac{\partial^2 \epsilon_{11}}{\partial X^2 \partial X^3} \right) = 0, \\ \Rightarrow & \frac{\partial^2 \epsilon_{12}}{\partial X^1 \partial X^3} + \frac{\partial^2 \epsilon_{31}}{\partial X^1 \partial X^2} = \frac{\partial^2 \epsilon_{11}}{\partial X^2 \partial X^3} + \frac{\partial^2 \epsilon_{32}}{\partial X_1^2}. \end{aligned} \quad (4)$$

$$p = 2 : i = 1, j = 3$$

$$\begin{aligned} & \varepsilon_{213} \left( \frac{\partial^2 \epsilon_{33}}{\partial X^1 \partial X^1} - \frac{\partial^2 \epsilon_{31}}{\partial X^3 \partial X^1} \right) + \varepsilon_{231} \left( \frac{\partial^2 \epsilon_{13}}{\partial X^2 \partial X^3} - \frac{\partial^2 \epsilon_{11}}{\partial X^3 \partial X^3} \right) = 0, \\ \Rightarrow & \frac{\partial^2 \epsilon_{13}}{\partial X^1 \partial X^3} + \frac{\partial^2 \epsilon_{31}}{\partial X^1 \partial X^3} = 2 \frac{\partial^2 \epsilon_{13}}{\partial X^1 \partial X^3} = \frac{\partial^2 \epsilon_{33}}{\partial X_1^2} + \frac{\partial^2 \epsilon_{11}}{\partial X_3^2}. \end{aligned} \quad (5)$$

$$p = 2 : i = 2, j = 3$$

$$\begin{aligned} & \varepsilon_{213} \left( \frac{\partial^2 \epsilon_{33}}{\partial X^2 \partial X^1} - \frac{\partial^2 \epsilon_{32}}{\partial X^3 \partial X^1} \right) + \varepsilon_{231} \left( \frac{\partial^2 \epsilon_{13}}{\partial X^2 \partial X^3} - \frac{\partial^2 \epsilon_{12}}{\partial X^3 \partial X^3} \right) = 0, \\ \Rightarrow & \frac{\partial^2 \epsilon_{13}}{\partial X^2 \partial X^3} + \frac{\partial^2 \epsilon_{32}}{\partial X^1 \partial X^3} = \frac{\partial^2 \epsilon_{33}}{\partial X^1 \partial X^2} + \frac{\partial^2 \epsilon_{12}}{\partial (X^3)^2}. \end{aligned} \quad (6)$$

$$p = 3 : i = 1, j = 2$$

$$\begin{aligned} & \varepsilon_{312} \left( \frac{\partial^2 \epsilon_{22}}{\partial X^1 \partial X^1} - \frac{\partial^2 \epsilon_{21}}{\partial X^2 \partial X^1} \right) + \varepsilon_{321} \left( \frac{\partial^2 \epsilon_{12}}{\partial X^1 \partial X^2} - \frac{\partial^2 \epsilon_{11}}{\partial X^2 \partial X^2} \right) = 0, \\ \Rightarrow & \frac{\partial^2 \epsilon_{22}}{\partial (X^1)^2} + \frac{\partial^2 \epsilon_{11}}{\partial (X^2)^2} = 2 \frac{\partial^2 \epsilon_{12}}{\partial X^1 \partial X^2}. \end{aligned} \quad (7)$$

$$p = 3 : i = 1, j = 3$$

$$\begin{aligned} & \varepsilon_{312} \left( \frac{\partial^2 \epsilon_{23}}{\partial X^1 \partial X^1} - \frac{\partial^2 \epsilon_{21}}{\partial X^3 \partial X^1} \right) + \varepsilon_{321} \left( \frac{\partial^2 \epsilon_{13}}{\partial X^1 \partial X^2} - \frac{\partial^2 \epsilon_{11}}{\partial X^3 \partial X^2} \right) = 0, \\ \Rightarrow & \frac{\partial^2 \epsilon_{23}}{\partial (X^1)^2} + \frac{\partial^2 \epsilon_{11}}{\partial X^2 \partial X^3} = \frac{\partial^2 \epsilon_{21}}{\partial X^1 \partial X^3} + \frac{\partial^2 \epsilon_{13}}{\partial X^1 \partial X^2}. \end{aligned} \quad (8)$$

$p = 3 : i = 2, j = 3$

$$\begin{aligned} & \varepsilon_{312} \left( \frac{\partial^2 \epsilon_{23}}{\partial X^2 \partial X^1} - \frac{\partial^2 \epsilon_{22}}{\partial X^3 \partial X^1} \right) + \varepsilon_{321} \left( \frac{\partial^2 \epsilon_{13}}{\partial X^2 \partial X^2} - \frac{\partial^2 \epsilon_{12}}{\partial X^3 \partial X^2} \right) = 0, \\ \Rightarrow & \frac{\partial^2 \epsilon_{23}}{\partial X^1 \partial X^2} + \frac{\partial^2 \epsilon_{12}}{\partial X^2 \partial X^3} = \frac{\partial^2 \epsilon_{22}}{\partial X^1 \partial X^3} + \frac{\partial^2 \epsilon_{13}}{\partial (X^2)^2}. \end{aligned} \quad (9)$$

Now, equations (3), (5) and (7) are independent, while (2) = (6), (4) = (8), and (1) = (9). Thus, there are six independent equation in total.

Now, given  $X^1 \rightarrow x$ ,  $X^2 \rightarrow y$ ,  $X^3 \rightarrow z$ , then:

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}, \quad (10)$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} = 2 \frac{\partial^2 \epsilon_{yz}}{\partial y \partial z}, \quad (11)$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} = 2 \frac{\partial^2 \epsilon_{zx}}{\partial z \partial x}, \quad (12)$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z} \left( \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} - \frac{\partial \epsilon_{xy}}{\partial z} \right), \quad (13)$$

$$\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x} \left( \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} - \frac{\partial \epsilon_{yz}}{\partial x} \right), \quad (14)$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{zx}}{\partial y} \right). \quad (15)$$

Note that (11) and (12) can be obtained by permuting suffices from (10); and (14), (15) can be obtained by permuting suffices from (13).

- (c). We found in (a) that  $\epsilon_{xz} = \epsilon_{yz} = 0$ ,  $\epsilon_{xy} = \frac{k(1+\nu)}{2E} [4\nu xy - x^2 - y^2]$  and  $\epsilon_{zz} = \frac{1-2\nu}{E} kxy$ , so that from (13) we find

$$\frac{1-2\nu}{E} k = 0,$$

which clearly is not true in general. Therefore, the stress state is not admissible.

## Question 5:

- (a) If  $\mathcal{C} = na_1 + ma_2$ , then

$$|C| = 2.461 \sqrt{n^2 + nm + m^2} \text{ \AA}, \quad r = \frac{|C|}{2\pi} \text{ \AA}, \quad \phi = \sin^{-1} \frac{\sqrt{3}m}{2\sqrt{n^2 + mn + m^2}}.$$

Thus,

(i)  $\mathcal{C}_1 = 3a_1 \Rightarrow n = 3, m = 0$ . Thus

$$|C| = 2.461\sqrt{3^2 + 0 \times 3 + 0^2} = 7.383 \text{ \AA},$$

$$r = \frac{7.383}{2\pi} = 1.175 \text{ \AA},$$

$$\phi = \sin^{-1} \frac{\sqrt{3} \times 0}{2\sqrt{3^2 + 0 \times 3 + 0}} = 0^\circ.$$

(ii)  $\mathcal{C}_2 = 3a_1 + 3a_2 \Rightarrow n = 3, m = 3$ . Thus

$$|C| = 2.461\sqrt{3^2 + 3 \times 3 + 3^2} = 12.788 \text{ \AA},$$

$$r = \frac{12.788}{2\pi} = 2.035 \text{ \AA},$$

$$\phi = \sin^{-1} \frac{\sqrt{3} \times 3}{2\sqrt{3^2 + 3 \times 3 + 3^2}} = 30^\circ.$$

(iii)  $\mathcal{C}_3 = 3a_1 + a_2 \Rightarrow n = 3, m = 1$ . Thus

$$|C| = 2.461\sqrt{3^2 + 1 \times 3 + 1^2} = 8.873 \text{ \AA},$$

$$r = \frac{8.873}{2\pi} = 1.412 \text{ \AA},$$

$$\phi = \sin^{-1} \frac{\sqrt{3} \times 1}{2\sqrt{3^2 + 1 \times 3 + 1^2}} = 13.898^\circ.$$

(iv)  $\mathcal{C}_4 = 4a_1 + a_2 \Rightarrow n = 4, m = 1$ . Thus

$$|C| = 2.461\sqrt{4^2 + 1 \times 4 + 1^2} = 11.278 \text{ \AA},$$

$$r = \frac{11.278}{2\pi} = 1.795 \text{ \AA},$$

$$\phi = \sin^{-1} \frac{\sqrt{3} \times 1}{2\sqrt{4^2 + 1 \times 4 + 1^2}} = 10.893^\circ.$$

(b)  $\mathcal{C}_1 = 3a_1$ :

- $\phi = 0^\circ \Rightarrow$  zig-zag
- $n - m = 3i$  for  $i = 1 \Rightarrow$  metallic  $\Rightarrow$  conductor
- 

$\mathcal{C}_1 = 3a_1 + 3a_2$ :

- $\phi = 30^\circ \Rightarrow$  armchair
- $n - m = 3i$  for  $i = 0 \Rightarrow$  conductor
-

$$\mathcal{C}_1 = 3a_1 + a_2:$$

- $\phi = 13.898^\circ \Rightarrow$  chiral
- $n - m \neq 3i \Rightarrow$  semi-conductor
- 

$$\mathcal{C}_1 = 4a_1 + a_2:$$

- $\phi = 10.893^\circ \Rightarrow$  chiral
- $n - m = 3i$  for  $i = 1 \Rightarrow$  conductor
- 

- (c) For a  $C_{70}$  fullerene, there are 70 carbon atoms  $\Rightarrow V = 70$ . As the  $C_{70}$  is a fullerene, then there are exactly 12 pentagons, so that if  $h$  denotes how many hexagons, then

$$F = 12 + h.$$

Next, as pentagons have 5 edges each and hexagons have 6 edges each, but each edge is shared exactly twice, then

$$E = \frac{5 \times 12 + 6h}{2} = 30 + 3h.$$

Thus, from Euler's theorem, we find

$$\begin{aligned} & F - E + V = 2, \\ \Rightarrow & 12 + h - 30 - 3h + 70 = 2, \\ \Rightarrow & 2h = 50, \\ \text{i.e.,} & h = 25. \end{aligned}$$

Therefore, there are exactly 25 hexagons in the  $C_{70}$  fullerene.