Family Name ....

First Name ....

Student Number ....

UNIVERSITY OF WOLLONGONG
SCHOOL OF MATHEMATICS AND APPLIED STATISTICS

MATH312 – Applied Mathematical Modelling III

Autumn Session 2007

Final Exam

Time Allowed: 3 hours and 15 minutes
Number of questions: 5.

DIRECTIONS TO CANDIDATES

1. Each question is to be attempted.
2. Each question is worth equal marks.
3. The examination paper is printed on both sides.
4. One solution book is provided. All solutions are to be done in the solution book.
5. WORKING (including all necessary *reasoning*) is to be shown for all solutions.
6. All notation is as used in lectures.

Examination Materials/Aids Allowed

Non-alphanumeric calculators are permitted.
A one-page, A4-sized, single-sided summary sheet is permitted.

Examination Materials/Aids to be supplied

None.

This examination paper must NOT be removed from the examination room.

Question 1:

(a) Assuming that all quantities refer to Cartesian coordinates, prove the following identities

(i) $\underset{\sim}{u} \times (\underset{\sim}{v}\underset{\sim}{w}) = (\underset{\sim}{u} \times \underset{\sim}{v})\underset{\sim}{w}$,

(ii) $\text{curl}(f\underset{\sim}{u}) = (\underset{\sim}{\nabla} f) \times \underset{\sim}{u} + f(\underset{\sim}{\nabla} \times \underset{\sim}{u})$,

(iii) $\underset{\sim}{u}(\underset{\sim}{v} \cdot \underset{\sim}{w} \times \underset{\sim}{r}) - \underset{\sim}{v}(\underset{\sim}{u} \cdot \underset{\sim}{w} \times \underset{\sim}{r}) = -(\underset{\sim}{u} \times \underset{\sim}{v}) \times (\underset{\sim}{w} \times \underset{\sim}{r})$,

where $\underset{\sim}{u}$, $\underset{\sim}{v}$, $\underset{\sim}{w}$ and $\underset{\sim}{r}$ are vectors and f is a scalar.

(b) Let (X^1, X^2, X^3) denote an initial curvilinear coordinate system. This coordinate system is then transformed into (x^1, x^2, x^3) such that the angles between the coordinate axes are given by

	X^1	X^2	X^3
x^1	90°	45°	135°
x^2	45°	60°	60°
x^3	45°	120°	120°

(i) If $x^i = \alpha^i_j X^j$, determine the matrix of direction cosines α^i_j.

(ii) A point P has the coordinates $(0, 1, -1)$ in the (X^1, X^2, X^3) coordinate system. Find the coordinate of the point P in the deformed coordinate system (x^1, x^2, x^3), which is denoted by P'.

(iii) On the same graph, show the (X^1, X^2, X^3) and (x^1, x^2, x^3) coordinate axes, and plot the points P and P'.

Question 2:

Let $\underset{\sim}{A}$ be a vector, which can be expressed as

$$\underset{\sim}{A} = 2\underset{\sim}{E}_1 - 3\underset{\sim}{E}_2 + 5\underset{\sim}{E}_3,$$

where $(\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3)$ are a set of tangential basis vectors in the directions of an orthogonal curvilinear coordinate system (X^1, X^2, X^3).

(a) If $\underset{\sim}{A}$ starts at the origin, sketch the geometrical representation of $\underset{\sim}{A}$.

(b) Describe in words the *movement* of $\underset{\sim}{A}$, from where it starts to where it finishes.

(c) Write down the contravariant components of $\underset{\sim}{A}$.

(d) Determine the contravariant and covariant physical components of $\underset{\sim}{A}$ in terms of basis vectors and the contravariant components of $\underset{\sim}{A}$.

(e) Now assume that the tangential basis vectors refer to Sally Forth's modified coordinate system, namely

$$x = \rho^2 \cos\vartheta, \qquad y = \rho^2 \sin\vartheta, \qquad z = \zeta.$$

Find the tangential basis vectors.

(f) Using 2(c) and 2(e) above, calculate the covariant components of $\underset{\sim}{A}$.

(g) Using 2(e) above, find the gradient basis vectors $(\underset{\sim}{E}^1, \underset{\sim}{E}^2, \underset{\sim}{E}^3)$.

(h) Express the physical components of $\underset{\sim}{A}$, found in 2(d), in terms of ρ, θ and ζ using the results from 2(e) and 2(g).

(i) What do the physical components found in 2(h) above represent?

(j) Check that the tangential and gradient basis vectors found in 2(e) and 2(g) above form a set of reciprocal bases.

Question 2(k) is for Honours students only.

(k) We know that the tangential and gradient basis vectors found in 2(e) and 2(g) above are related by $\underset{\sim}{E}_i = k_i \underset{\sim}{E}^i$, where k_i is a scalar. Find k_1, k_2 and k_3, and describe how they relate to components of the metric tensor.

Question 3:

Consider the deformation

$$x^1 = 2X^1 + X^2, \quad x^2 = X^1 + X^2, \quad x^3 = X^3,$$

where (X^1, X^2, X^3) refers to an orthogonal curvilinear coordinate system corresponding to the undeformed body of a continuum material, while (x^1, x^2, x^3) refers to another orthogonal curvilinear coordinate system corresponding to the deformed body of the continuum material.

(a) Find the deformation matrix $\mathbf{F}$.

(b) The deformation matrix can be expressed as either

$$\mathbf{F} = \mathbf{RU} \quad \text{or} \quad \mathbf{F} = \mathbf{VR},$$

where $\mathbf{R}$ is the rotation matrix and $\mathbf{U}$ and $\mathbf{V}$ are the symmetric matrices usually referred to as the right and left stretch tensors respectively. Show that

(i) $\mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2}$, and

(ii) $\mathbf{V} = (\mathbf{F}\mathbf{F}^T)^{1/2}$.

(c) Determine $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.

If you are an Honours student, for Question 3(c) you must also:

Show that the eigenvalues and corresponding eigenvectors of $\mathbf{C}$ are as given in 3(d).

(d) Assume that the eigenvalues and corresponding eigenvectors of $\mathbf{C}$ are

$$\lambda_1 = \frac{7}{2} + \frac{3\sqrt{5}}{2}, \quad \underset{\sim}{x}_1 = \left(1, -\frac{1}{2} + \frac{\sqrt{5}}{2}, 0\right)^T r,$$

$$\lambda_2 = \frac{7}{2} - \frac{3\sqrt{5}}{2}, \quad \underset{\sim}{x}_2 = \left(1, -\frac{1}{2} - \frac{\sqrt{5}}{2}, 0\right)^T s,$$

$$\lambda_3 = 1, \quad \underset{\sim}{x}_3 = (0, 0, 1)^T t,$$

where r, s and t are parameters. Then, let $\mathbf{D}$ and $\mathbf{P}$ be the matrices defined by

$$\mathbf{D} = \begin{pmatrix} \frac{7}{2} + \frac{3\sqrt{5}}{2} & 0 & 0 \\ 0 & \frac{7}{2} - \frac{3\sqrt{5}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & 0 \\ \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where these matrices satisfy

$$\mathbf{C} = \mathbf{PDP}^{-1}. \tag{1}$$

(i) Calculate the matrix product $\mathbf{PP}$.

(ii) From 3(di), write down $\mathbf{P}^{-1}$.

(iii) Explain why we consider the matrix decomposition given in (1), in relation to finding the right stretch tensor $\mathbf{U}$.

(e) Using 3(d), or otherwise, find $\sqrt{\mathbf{C}}$, and therefore write down the right stretch tensor $\mathbf{U}$.

(f) Using 3(e), or otherwise, find the rotation matrix $\mathbf{R}$ and the left stretch tensor $\mathbf{V}$. Upon examining $\mathbf{R}$ and $\mathbf{V}$, what can can you say about the deformation?

(g) Assume the deformation is applied to a unit cube of material. On the same graph, sketch the undeformed and deformed bodies of the continuum material. Give a brief description about the resulting deformed shape in comparison to the initial undeformed shape.

Question 4:

In Cartesian coordinates (x, y, z), a particular stress state, σ_{ij}, is given by

$$\begin{aligned}
\sigma_{xx} &= k[xy - \nu(x^2 - y^2)], \\
\sigma_{yy} &= k[yx - \nu(y^2 - x^2)], \\
\sigma_{zz} &= kxy, \\
\sigma_{xy} &= \frac{k}{2}[4\nu xy - x^2 - y^2], \\
\sigma_{xz} &= \sigma_{yz} = 0,
\end{aligned}$$

where k is a non-zero constant and ν is Poisson's ratio.

(a) For this stress state, determine the corresponding strain components, ϵ_{ij}, according to the generalized Hooke's Law for isotropic materials.

(b) To uniquely determine the strains, consider the compatibility conditions given by

$$\varepsilon_{pks}(\epsilon_{sj,ik} - \epsilon_{si,jk}) = 0.$$

Show that the six independent compatibility equations in Cartesian coordinates are

$$\begin{aligned}
\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} &= 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}, \\
\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} &= 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z}, \\
\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} &= 2\frac{\partial^2 \epsilon_{zx}}{\partial z \partial x}, \\
\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} - \frac{\partial \epsilon_{xy}}{\partial z}\right), \\
\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} &= \frac{\partial}{\partial x}\left(\frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} - \frac{\partial \epsilon_{yz}}{\partial x}\right), \\
\frac{\partial^2 \epsilon_{yy}}{\partial z \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{zx}}{\partial y}\right).
\end{aligned}$$

(c) Hence, or otherwise, determine if this is an admissible state of stress for a continuum.

Question 4(d) is for Honours students only

(d) In a short paragraph, explain why we need to consider the compatibility equations for a linear elastic isotropic continuum material.

Question 5:

(a) For the carbon nanotubes with the following chiral vectors $\mathcal{C}_i = na_1 + ma_2$, for $i = 1, \ldots, 4$, determine the radius, circumference and chiral angle.

(i) $\mathcal{C}_1 = 3a_1$.

(ii) $\mathcal{C}_2 = 3a_1 + 3a_2$.

(iii) $\mathcal{C}_3 = 3a_1 + a_2$.

(iv) $\mathcal{C}_4 = 4a_1 + a_2$.

(b) For each of the carbon nanotubes with chiral vector given in 5(a):

(i) State whether they are of the type zig-zag, armchair, or chiral.

(ii) Determine whether the nanotube is a conductor or semi-conductor of electricity.

(iii) Sketch the appropriate graphene sheet with chiral vector being horizontal, and note the location and magnitude of the chiral angle on your sketch.

(c) Consider the C_{70} fullerene, whose geometrical structure satisfies Euler's theorem

$$F - E + V = 2,$$

where F, E and V are the number of faces, edges and vertices of the fullerene, respectively. Thus, given that a fullerene only contains pentagons and hexagons, show there are exactly 25 hexagons in a C_{70} fullerene.

Question 5(d) is for Honours students only

(d) For a fullerene with only pentagons and hexagons, show that there must be exactly 12 pentagons. What does this say about the minimum size of a fullerene?

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