Review

In the previous lecture, we ...

- considered the physical components of tensors
- introduced the tangential and gradient basis vectors

Aims

In this lecture, we will ...

- examine the meaning of physical components
- consider the physical components of higher order tensors

2.6.3 What components mean - summary

In Cartesian coordinates (x, y, z), the set of basis vectors $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ are orthonormal, so that the components of a vector $\underset{\sim}{A}$, say

$$\underset{\sim}{A} = A_x \underset{\sim}{e}_1 + A_y \underset{\sim}{e}_2 + A_z \underset{\sim}{e}_3,$$

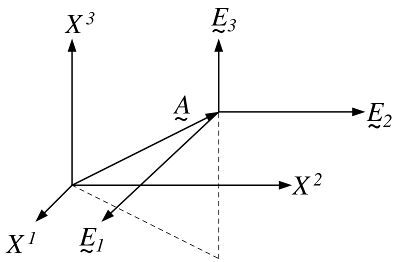
define how many units of length the vector *moves* along in the respective directions.

For example, if

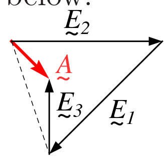
$$\underset{\sim}{A} = 2\underset{\sim}{e}_1 + \underset{\sim}{e}_2 - 3\underset{\sim}{e}_3,$$

then the vector *moves* along 2 units in the z^1 direction, 1 unit in the z^2 direction and -3 units in the z^3 direction.

In any general curvilinear coordinates system, say (X^1, X^2, X^3), the set of tangential basis vectors $(\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3)$ define three vectors in the directions of X^1, X^2, X^3 respectively, but they need not be orthogonal or unit vectors, i.e., $|\underset{\sim}{E}_i| \neq 1$.



If $\underset{\sim}{A} = \underset{\sim}{E}_1 + \underset{\sim}{E}_2 + \underset{\sim}{E}_3$, then the direction of $\underset{\sim}{A}$ is shown in red below:



Thus, if

$$\underset{\sim}{A} = 3\underset{\sim}{E}_1 - 2\underset{\sim}{E}_2 + 42\underset{\sim}{E}_3,$$

then it means that the vector *moves* along 3 lengths of $\underset{\sim}{E}_1$ in the X^1 direction, -2 lengths of $\underset{\sim}{E}_2$ in the X^2 direction and 42 lengths of $\underset{\sim}{E}_3$ units in the X^3 direction.

Hence, if

$$\underset{\sim}{A} = A^1\underset{\sim}{E}_1 + A^2\underset{\sim}{E}_2 + A^3\underset{\sim}{E}_3,$$

then the contravariant components A^i represents how many lengths of $\underset{\sim}{E}_i$ that the vector *moves* in the X^i direction, and is given by $A^i = \underset{\sim}{A} \cdot \underset{\sim}{E}^i$, where $\underset{\sim}{E}^i$ is the corresponding gradient basis vector.

But the question is, how many units of length have we moved?

To consider the physical components of a vector $\underset{\sim}{A}$, consider

$$\begin{aligned}\underset{\sim}{A} &= A^1\underset{\sim}{E}_1 + A^2\underset{\sim}{E}_2 + A^3\underset{\sim}{E}_3,\\ &= A^1|\underset{\sim}{E}_1|\frac{\underset{\sim}{E}_1}{|\underset{\sim}{E}_1|} + A^2|\underset{\sim}{E}_2|\frac{\underset{\sim}{E}_2}{|\underset{\sim}{E}_2|} + A^3|\underset{\sim}{E}_3|\frac{\underset{\sim}{E}_3}{|\underset{\sim}{E}_3|},\\ &= A^{(1)}\frac{\underset{\sim}{E}_1}{|\underset{\sim}{E}_1|} + A^{(2)}\frac{\underset{\sim}{E}_2}{|\underset{\sim}{E}_2|} + A^{(3)}\frac{\underset{\sim}{E}_3}{|\underset{\sim}{E}_3|}.\end{aligned}$$

Thus, the physical component $A^{(i)}$ tells us how many units of length we *move* along in the X^i direction ($= \underset{\sim}{E}_i$ direction).

Note:

From above, we find $A^{(1)} = A^1|\underset{\sim}{E}_1|$, which means the physical component $A^{(1)}$ is actually a scalar. This contraction is required for the indices of the terms in the above equations to *balance*.

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Note:

$$A^{(i)} = A^{\underline{i}}|\underset{\sim}{E}_{\underline{i}}| \qquad \Longleftrightarrow \qquad A^i = \frac{A^{(\underline{i})}}{|\underset{\sim}{E}_{\underline{i}}|}.$$

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Now, recall

$$g_{ij} = \underset{\sim}{E}_i \cdot \underset{\sim}{E}_j,$$

so that if we let $j = i$, then

$$g_{\underline{i}\,\underline{i}} = \underset{\sim}{E}_{\underline{i}} \cdot \underset{\sim}{E}_{\underline{i}} = |\underset{\sim}{E}_{\underline{i}}|^2.$$

Therefore, we have

$$A^{(i)} = A^i\sqrt{g_{ii}} \qquad \Longleftrightarrow \qquad A^i = \frac{A^{(i)}}{\sqrt{g_{ii}}},$$

where g_{ij} is found from the coordinate transformation relations

$$z^1 = z^1(X^1, X^2, X^3), \quad z^2 = z^2(X^1, X^2, X^3), \quad z^3 = z^3(X^1, X^2, X^3).$$

Similarly, we can show

$$A_{(i)} = A_i\sqrt{g^{ii}} \qquad \Longleftrightarrow \qquad A_i = \frac{A_{(i)}}{\sqrt{g^{ii}}}.$$

Note:

We know

$$A^{(i)} = A^i\sqrt{g_{ii}} \quad \text{and} \quad A_{(i)} = A_i\sqrt{g^{ii}}$$

are a scalars. This implies that:

1. $\sqrt{g_{\underline{i}\,\underline{i}}}$ is a covariant tensor of order 1, and
2. $\sqrt{g^{\underline{i}\,\underline{i}}}$ is a contravariant tensor of order 1.

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Example 2.23:

In cylindrical polar coordinates (r, θ, z), the metric tensor is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the physical components $A^{(i)}$, if

$$\underset{\sim}{A} = A^r\underset{\sim}{E}_r + A^\theta\underset{\sim}{E}_\theta + A^z\underset{\sim}{E}_z.$$

✠

Answer:

In cylindrical polar coordinates, the tangential base vectors $\underset{\sim}{E}_i$ are orthogonal. And, as $A^{(i)} = A^i\sqrt{g_{ii}}$, then

$$A^{(1)} = A^1\sqrt{g_{11}} = A^r\sqrt{1} = A^r,$$
$$A^{(2)} = A^2\sqrt{g_{22}} = A^\theta\sqrt{r^2} = rA^\theta,$$
$$A^{(3)} = A^3\sqrt{g_{33}} = A^z\sqrt{1} = A^z.$$

□

Note:

From our two definitions for the physical components, we can show

$$A^{(i)} = \frac{A_{\underline{i}}}{|\underset{\sim}{E}_{\underline{i}}|} \quad \& \quad A^{(i)} = A^{\underline{i}}\,|\underset{\sim}{E}_{\underline{i}}| \quad \Rightarrow \quad \frac{A_{\underline{i}}}{A^{\underline{i}}} = |\underset{\sim}{E}_i|^2 = g_{\underline{i}\,\underline{i}}\,.$$

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Exercise 2.22:

Given Example 2.23, find the physical components $A_{(i)}$.

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Answer:

□

Note:

1. Because the cylindrical polar coordinates are orthogonal, the gradient basis vectors and the tangential basis vectors are respectively parallel, so that the contravariant and covariant physical components are identical.

Note:

2. All the physical components have the same dimensions, e.g.,

$$A^{(1)} = A_{(1)} = A^r \quad \Rightarrow \quad [A^r] = \text{length},$$
$$A^{(2)} = A_{(2)} = rA^\theta \quad \Rightarrow \quad [rA^\theta] = \text{length}, \quad ([A^\theta] = 1)$$
$$A^{(3)} = A_{(3)} = A^z \quad \Rightarrow \quad [A^z] = \text{length}.$$

3. Therefore, vectors in terms of physical components are in *agreement* with the physical world.

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2.6.4 Higher order physical components

The physical components can be extended to higher orders. For instance, for orthogonal curvilinear coordinates, consider the second-order tensor t^i_j

$$t^i = t^i_j n^j,$$

where t^i and n^j are contravariant vectors.

If we now replace t^i and n^j with their expressions in terms of their physical components, namely

$$\frac{t^{(i)}}{\sqrt{g_{ii}}} = \sum_j t^i_j \frac{n^{(j)}}{\sqrt{g_{jj}}},$$

$$\Rightarrow \qquad t^{(i)} = \sum_j \frac{\sqrt{g_{ii}}}{\sqrt{g_{jj}}} t^i_j n^{(j)} = t^{(i)}_{(j)} n^{(j)}.$$

Thus, the *second-order physical components* are defined by

$$t^{(i)}_{(j)} = \sqrt{\frac{g_{ii}}{g_{jj}}}\, t^i_j \qquad \Longleftrightarrow \qquad t^i_j = \sqrt{\frac{g_{jj}}{g_{ii}}}\, t^{(i)}_{(j)}.$$

By raising and lower indices, we have

$$t_{kj} = g_{ik} t^i_j \qquad \Rightarrow \qquad t_{kj} = g_{ik}\sqrt{\frac{g_{jj}}{g_{ii}}}\, t^{(i)}_{(j)}, \qquad \text{(summing } i\text{)}$$

$$t^{ki} = g^{jk} t^i_j \qquad \Rightarrow \qquad t^{ki} = g^{jk}\sqrt{\frac{g_{jj}}{g_{ii}}}\, t^{(i)}_{(j)}. \qquad \text{(summing } j\text{)}$$

Recall, that for orthogonal curvilinear coordinates,

$$g_{ik} = \underset{\sim}{E}_i \cdot \underset{\sim}{E}_k = \underset{\sim}{E}_i \cdot \underset{\sim}{E}_i \delta_{ik} = g_{ii}\delta_{ik},$$

so that

$$t_{kj} = g_{kk}\sqrt{\frac{g_{jj}}{g_{kk}}}\, t^{(k)}_{(j)} = \sqrt{g_{kk}g_{jj}}\, t^{(k)}_{(j)}.$$

Similarly, we can show that

$$t^{ki} = g^{kk}\sqrt{\frac{g_{kk}}{g_{ii}}}\, t^{(i)}_{(k)} = \frac{1}{g_{kk}}\sqrt{\frac{g_{kk}}{g_{ii}}}\, t^{(i)}_{(k)} = \frac{1}{\sqrt{g_{ii}g_{kk}}}\, t^{(i)}_{(k)}.$$

Finally, upon solving for $t^{(i)}_{(j)}$ in the above and comparing, we obtain the relationship

$$t^{(i)}_{(j)} = \sqrt{\frac{g_{ii}}{g_{jj}}}\, t^i_j = \sqrt{g_{ii}g_{jj}}\, t^{ij} = \frac{1}{\sqrt{g_{ii}g_{jj}}}\, t_{ij}.$$

Exercise 2.23:

For cylindrical polar coordinates (r, θ, z), the metric tensor is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and let t^i_j have the following physical components

$$t^{(i)}_{(j)} = \begin{pmatrix} 1 & -r & 0 \\ r & \frac{1}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find t^i_j, and the associated tensors t^{ij} and t_{ij}.

✠

Answer:

□

Question?

What is wrong with Exercise 2.23?

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Answer:

□

Summary

In this lecture, we . . .

- examine the meaning of physical components
- considered the physical components of higher order tensors

Coming up

In the next lecture, we . . .

- introduce direction cosines
- introduce dyads

Homework Exercise 2.9:

1. The metric tensor in spherical polar coordinates (r, θ, ϕ) is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}.$$

   Find the physical components $A^{(i)}$ and $A_{(i)}$ of

$$\underset{\sim}{A} = A^r \underset{\sim}{E}_r + A^\theta \underset{\sim}{E}_\theta + A^\phi \underset{\sim}{E}_\phi.$$

2. From 1., let

$$t^{(i)}_{(j)} = A^{(i)} A_{(j)}.$$

   Find the tensors t_{ij}, t^i_j and t^{ij}.