Review

In the previous lecture, we ...

- considered covariant/contravariant tensors of order n
- examined how to express the metric tensor in terms of base vectors

Aims

In this lecture, we will ...

- consider the physical components of tensors
- introduce the tangential and gradient basis vectors

2.6.2 Physical components of a tensor

Consider the metric tensors for the usual Cartesian and cylindrical polar coordinates, namely

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively.

Question?

Are the units of a typical element of g_{ij} always the same?

✠

Answer:

□

This result is in contradiction of the physical world. For example, consider a physical velocity vector $\underset{\sim}{V} = (V^1, V^2, V^3)$, where each component of the vector must have the same units, e.g., ms^{-1}, regardless of the coordinate system.

This raises the idea of the physical dimensions of the components of a tensor, or more commonly called the *physical components* of a tensor, in a specific direction.

Definition 2.10:

The *physical components* of the vector $\underset{\sim}{A}$ in a direction is defined as the projection of $\underset{\sim}{A}$ upon a unit vector in the desired direction.

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Note:

We denote the ith physical contravariant or covariant components of a vector $\underset{\sim}{A}$ in the direction of the ith base vector by $A^{(i)}$ or $A_{(i)}$, respectively.

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Exercise 2.18:

In the usual Cartesian coordinates (x, y, z) with usual orthonormal base vectors $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$, let the vector $\underset{\sim}{A}$ be expressed by

$$\underset{\sim}{A} = A_x \underset{\sim}{e}_1 + A_y \underset{\sim}{e}_2 + A_z \underset{\sim}{e}_3.$$

Write down the physical components of $\underset{\sim}{A}$ in the direction of:

1. $\underset{\sim}{e}_1$
2. $\underset{\sim}{e}_2$
3. $\underset{\sim}{e}_3$

✠

Answer:

□

Note:

1. In Exercise 2.18, we expressed the vector $\underset{\sim}{A}$ in terms of orthonormal base vectors, which meant the base vectors are also unit vectors. Hence, finding the physical components along the directions of the base vectors was easy.
2. If a tensor is expressed in reference to unit base vectors, then the components of the tensor are equal to the physical components (along the base vector directions).
3. As most Cartesian tensors refer to the usual orthonormal base vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$, then they are physical tensors.

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This raises the question of how to find the physical components for *any* tensor.

To answer this question, lets begin by just considering tensors of order 1, i.e., vectors.

We have seen previously that an arbitrary vector $\underset{\sim}{A}$ can be represented in many forms - depending on the coordinate system and basis vectors selected.

For example, in Cartesian coordinates, we can represent $\underset{\sim}{A}$ as

$$\underset{\sim}{A} = A_x \underset{\sim}{e}_1 + A_y \underset{\sim}{e}_2 + A_z \underset{\sim}{e}_3,$$

where $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ are the usual basis vectors and A_x, A_y, A_z are the respective components.

Further, we can represent $\underset{\sim}{A}$ in any general curvilinear coordinate system, say (x^1, x^2, x^3), in terms of either contravariant components (A^1, A^2, A^3) or covariant components (A_1, A_2, A_3).

For example,

$$\underset{\sim}{A} = A^1 \underset{\sim}{E}_1 + A^2 \underset{\sim}{E}_2 + A^3 \underset{\sim}{E}_3,$$

or,

$$\underset{\sim}{A} = A_1 \underset{\sim}{E}^1 + A_2 \underset{\sim}{E}^2 + A_3 \underset{\sim}{E}^3,$$

where $(\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3)$ and $(\underset{\sim}{E}^1, \underset{\sim}{E}^2, \underset{\sim}{E}^3)$ represent a set of tangential basis vectors and a set of gradient basis vectors, respectively.

Note:

1. Tangential basis vectors are parallel to the coordinate axes at a point.
2. Gradient basis vectors are perpendicular to the coordinate surface at a point.
3. For an orthogonal basis, the tangential and gradient basis may be the same.
4. $(\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3)$ and $(\underset{\sim}{E}^1, \underset{\sim}{E}^2, \underset{\sim}{E}^3)$ are a set of reciprocal bases, i.e.,
$$\underset{\sim}{E}^i \cdot \underset{\sim}{E}_j = \delta^i_j.$$
5. These vector expansions are simply different ways of representing the same vector. In general, the physical dimensions of the components A^i and A_j are not the same.

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Recalling that the dot product can be used to find the angle between two vectors (see MATH187 notes), then the necessary relation of

$$\underset{\sim}{E}^i \cdot \underset{\sim}{E}_j = \delta^i_j,$$

implies that:

1. $\underset{\sim}{E}^1 \perp \underset{\sim}{E}_2$ and $\underset{\sim}{E}^1 \perp \underset{\sim}{E}_3 \quad \Rightarrow \quad \underset{\sim}{E}^1 = \frac{1}{V}(\underset{\sim}{E}_2 \times \underset{\sim}{E}_3)$,
2. $\underset{\sim}{E}^2 \perp \underset{\sim}{E}_1$ and $\underset{\sim}{E}^2 \perp \underset{\sim}{E}_3 \quad \Rightarrow \quad \underset{\sim}{E}^2 = \frac{1}{V}(\underset{\sim}{E}_3 \times \underset{\sim}{E}_1)$,
3. $\underset{\sim}{E}^3 \perp \underset{\sim}{E}_1$ and $\underset{\sim}{E}^3 \perp \underset{\sim}{E}_2 \quad \Rightarrow \quad \underset{\sim}{E}^3 = \frac{1}{V}(\underset{\sim}{E}_1 \times \underset{\sim}{E}_2)$,

where

$$V = \underset{\sim}{E}_1 \cdot (\underset{\sim}{E}_2 \times \underset{\sim}{E}_3).$$

Note:

The cyclic order of the indices above must be maintained.

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Exercise 2.19:

Let $P(x, y, z)$ be a point in $\mathbb{R}^3$, and assume the usual orthonormal Cartesian base vectors

$$\underset{\sim}{e}_1 = \underset{\sim}{i}\,, \qquad \underset{\sim}{e}_2 = \underset{\sim}{j}\,, \qquad \underset{\sim}{e}_3 = \underset{\sim}{k}.$$

At the point P:

1. Write down a set of tangential base vectors $(\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3)$.
2. Determine the set of corresponding gradient base vectors $(\underset{\sim}{E}^1, \underset{\sim}{E}^2, \underset{\sim}{E}^3)$.
3. Check that the reciprocal basis condition holds true, i.e.,
$$\underset{\sim}{E}^i \cdot \underset{\sim}{E}_j = \delta^i_j.$$

✠

Answer:

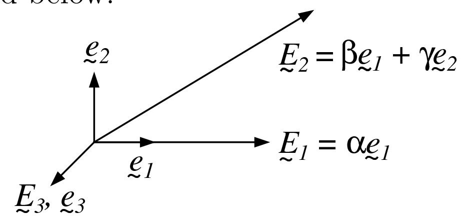
□

Example 2.22:

Let α, β, γ denote nonzero positive constants such that $\alpha\gamma = 1$ is satisfied. Consider the nonorthogonal tangential basis vectors

$$\utilde{E}_1 = \alpha \utilde{e}_1, \qquad \utilde{E}_2 = \beta \utilde{e}_1 + \gamma \utilde{e}_2, \qquad \utilde{E}_3 = \utilde{e}_3,$$

and are illustrated below:



The reciprocal basis can be shown (see Homework Exercise 2.8) to be

$$\utilde{E}^1 = \gamma \utilde{e}_1 - \beta \utilde{e}_2, \qquad \utilde{E}^2 = \alpha \utilde{e}_2, \qquad \utilde{E}^3 = \utilde{e}_3.$$

□

Now, recall the three representations of a vector $\utilde{A}$, namely

$$\utilde{A} = \begin{cases} A_x \utilde{e}_1 + A_y \utilde{e}_2 + A_z \utilde{e}_3, \\ A^1 \utilde{E}_1 + A^2 \utilde{E}_2 + A^3 \utilde{E}_3, \\ A_1 \utilde{E}^1 + A_2 \utilde{E}^2 + A_3 \utilde{E}^3. \end{cases}$$

While we now know about the tangential and gradient basis vectors, the question still remains how do we find the physical components of $\utilde{A}$ in a particular direction?

From Definition 2.10, we said that the physical components of a tensor are equal to the projection onto a unit vector in the appropriate directions.

Thus, the physical components in the direction of $\utilde{E}^1$, $\utilde{E}^2$ and $\utilde{E}^3$ are respectively given by

$$A_{(1)} = \utilde{A} \cdot \frac{\utilde{E}^1}{|\utilde{E}^1|} = (A^1 \utilde{E}_1 + A^2 \utilde{E}_2 + A^3 \utilde{E}_3) \cdot \frac{\utilde{E}^1}{|\utilde{E}^1|} = \frac{A^1}{|\utilde{E}^1|},$$
$$A_{(2)} = \utilde{A} \cdot \frac{\utilde{E}^2}{|\utilde{E}^2|} = (A^1 \utilde{E}_1 + A^2 \utilde{E}_2 + A^3 \utilde{E}_3) \cdot \frac{\utilde{E}^2}{|\utilde{E}^2|} = \frac{A^2}{|\utilde{E}^2|},$$
$$A_{(3)} = \utilde{A} \cdot \frac{\utilde{E}^3}{|\utilde{E}^3|} = (A^1 \utilde{E}_1 + A^2 \utilde{E}_2 + A^3 \utilde{E}_3) \cdot \frac{\utilde{E}^3}{|\utilde{E}^3|} = \frac{A^3}{|\utilde{E}^3|},$$

noting that we can use *any* valid representation for $\utilde{A}$ when preforming the projection.

For simplicity, we have used the contravariant representation because we can then use the relationship $\utilde{E}_i \cdot \utilde{E}^j = \delta_i^j$ to simplify the results.

Similarly, the physical components in the direction of $\utilde{E}_1$, $\utilde{E}_2$ and $\utilde{E}_3$ are respectively given by

$$A^{(1)} = \utilde{A} \cdot \frac{\utilde{E}_1}{|\utilde{E}_1|} = (A_1 \utilde{E}^1 + A_2 \utilde{E}^2 + A_3 \utilde{E}^3) \cdot \frac{\utilde{E}_1}{|\utilde{E}_1|} = \frac{A_1}{|\utilde{E}_1|},$$
$$A^{(2)} = \utilde{A} \cdot \frac{\utilde{E}_2}{|\utilde{E}_2|} = (A_1 \utilde{E}^1 + A_2 \utilde{E}^2 + A_3 \utilde{E}^3) \cdot \frac{\utilde{E}_2}{|\utilde{E}_2|} = \frac{A_2}{|\utilde{E}_2|},$$
$$A^{(3)} = \utilde{A} \cdot \frac{\utilde{E}_3}{|\utilde{E}_3|} = (A_1 \utilde{E}^1 + A_2 \utilde{E}^2 + A_3 \utilde{E}^3) \cdot \frac{\utilde{E}_3}{|\utilde{E}_3|} = \frac{A_3}{|\utilde{E}_3|}.$$

Note:

In the above examples, the values for $A_{(i)}$ and $A^{(i)}$ are different, namely $\frac{A^{\underline{i}}}{|\utilde{E}^{\underline{i}}|}$ or $\frac{A_{\underline{i}}}{|\utilde{E}_{\underline{i}}|}$, as they are the components of $\utilde{A}$ projected along different directions, either $\utilde{E}^i$ or $\utilde{E}_i$.

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Exercise 2.20:
From Example 2.22, we have

$$\underset{\sim}{E}_1 = \alpha \underset{\sim}{e}_1, \qquad \underset{\sim}{E}_2 = \beta \underset{\sim}{e}_1 + \gamma \underset{\sim}{e}_2, \qquad \underset{\sim}{E}_3 = \underset{\sim}{e}_3,$$
$$\underset{\sim}{E}^1 = \gamma \underset{\sim}{e}_1 - \beta \underset{\sim}{e}_2, \qquad \underset{\sim}{E}^2 = \alpha \underset{\sim}{e}_2, \qquad \underset{\sim}{E}^3 = \underset{\sim}{e}_3.$$

Find the physical components along $\underset{\sim}{E}^1$, $\underset{\sim}{E}^2$, $\underset{\sim}{E}_1$ and $\underset{\sim}{E}_2$.

✠

Answer:

□

Note:
We will obtain the same values for the physical components if we used a different vector representation for $\underset{\sim}{A}$, i.e.,

$$A^{(2)} = \underset{\sim}{A} \cdot \frac{\underset{\sim}{E}_2}{|\underset{\sim}{E}_2|} = (A^1 \underset{\sim}{E}_1 + A^2 \underset{\sim}{E}_2 + A^3 \underset{\sim}{E}_3) \cdot \frac{\underset{\sim}{E}_2}{|\underset{\sim}{E}_2|}$$
$$= \frac{A^1(\underset{\sim}{E}_1 \cdot \underset{\sim}{E}_2) + A^2(\underset{\sim}{E}_2 \cdot \underset{\sim}{E}_2) + A^3(\underset{\sim}{E}_3 \cdot \underset{\sim}{E}_2)}{|\underset{\sim}{E}_2|}.$$

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Question?

Given that

$$\underset{\sim}{A} = A_x \underset{\sim}{e}_1 + A_y \underset{\sim}{e}_2 + A_z \underset{\sim}{e}_3,$$

then how can we find the contravariant and covariant components of the vector $\underset{\sim}{A}$?

✠

Answer:

□

(c.f. The physical components of $\underset{\sim}{A}$ are projected onto the appropriate *unit* vectors)

Exercise 2.21:

From Exercise 2.20, we have

$$\underset{\sim}{E}_1 = \alpha \underset{\sim}{e}_1, \qquad \underset{\sim}{E}_2 = \beta \underset{\sim}{e}_1 + \gamma \underset{\sim}{e}_2, \qquad \underset{\sim}{E}_3 = \underset{\sim}{e}_3,$$
$$\underset{\sim}{E}^1 = \gamma \underset{\sim}{e}_1 - \beta \underset{\sim}{e}_2, \qquad \underset{\sim}{E}^2 = \alpha \underset{\sim}{e}_2, \qquad \underset{\sim}{E}^3 = \underset{\sim}{e}_3.$$

Given that

$$\underset{\sim}{A} = A_x \underset{\sim}{e}_1 + A_y \underset{\sim}{e}_2 + A_z \underset{\sim}{e}_3,$$

then find the contravariant components A^1 and A^2 and the covariant components A_1 and A_2 in terms of A_x, A_y, A_z, α, β and γ.

Then, using your answers from Exercise 2.20, express the physical components $A^{(1)}$, $A^{(2)}$, $A_{(1)}$ and $A_{(2)}$ in terms of A_x, A_y, A_z, α, β and γ.

✠

Answer:

□

Summary

In this lecture, we ...

- considered the physical components of tensors
- introduced the tangential and gradient basis vectors

Coming up

In the next lecture, we ...

- consider the physical components of higher order tensors
- introduce direction cosines

Homework Exercise 2.8:

1. Assuming that $\alpha\gamma = 1$, show that

$$\underset{\sim}{E}_1 = \alpha \underset{\sim}{e}_1, \qquad \underset{\sim}{E}_2 = \beta \underset{\sim}{e}_1 + \gamma \underset{\sim}{e}_2, \qquad \underset{\sim}{E}_3 = \underset{\sim}{e}_3,$$
$$\underset{\sim}{E}^1 = \gamma \underset{\sim}{e}_1 - \beta \underset{\sim}{e}_2, \qquad \underset{\sim}{E}^2 = \alpha \underset{\sim}{e}_2, \qquad \underset{\sim}{E}^3 = \underset{\sim}{e}_3,$$

   are reciprocal bases.

2. Assume that $\underset{\sim}{A}$ can be expressed in Cartesian coordinates as

$$\underset{\sim}{A} = \alpha \underset{\sim}{e}_1 - \beta \underset{\sim}{e}_2 + \gamma \underset{\sim}{e}_3,$$

   and we have the nonorthogonal basis vectors

$$\underset{\sim}{E}_1 = 2\underset{\sim}{e}_1 - \underset{\sim}{e}_2, \qquad \underset{\sim}{E}_2 = \underset{\sim}{e}_2, \qquad \underset{\sim}{E}_3 = \underset{\sim}{e}_3.$$

   Express the physical components $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ in terms of α and β.