Review

In the previous lecture, we ...

- introduced the conjugate metric tensor
- introduced the concepts of contravariant and covariant tensors

Aims

In this lecture, we will ...

- consider covariant/contravariant tensors of order n
- examine how to express the metric tensor in terms of base vectors

2.4.1 Higher order tensors

We have shown previously that first order tensors are quantities that obey certain transformation laws. Higher order tensors are defined in a similar manner.

Definition 2.6:

Whenever the nine quantities A^{ij} in a coordinate system (x^1, x^2, x^3) are related to nine quantities $\bar{A}^{mn}$ in a coordinate system (X^1, X^2, X^3) such that the transformation law

$$\bar{A}^{mn}(\underset{\sim}{X}) = A^{ij}(\underset{\sim}{x})\frac{\partial X^m}{\partial x^i}\frac{\partial X^n}{\partial x^j}$$

is satisfied, then these quantities are called *components of a contravariant tensor of order two.*

□

Example 2.17:

An example of a contravariant tensor of order two is the conjugate metric tensor g^{jk}, i.e.,

$$\bar{g}^{mn} = g^{jk}\frac{\partial X^m}{\partial x^j}\frac{\partial X^n}{\partial x^k}.$$

□

Definition 2.7:

Whenever the nine quantities A_{ij} in a coordinate system (x^1, x^2, x^3) are related to nine quantities $\bar{A}_{mn}$ in a coordinate system $(X^1, X^2, , X^3)$ such that the transformation law

$$\bar{A}_{mn}(\underset{\sim}{X}) = A_{ij}(\underset{\sim}{x})\frac{\partial x^i}{\partial X^m}\frac{\partial x^j}{\partial X^n}$$

is satisfied, then these quantities are called *components of a covariant tensor of order two.*

□

Example 2.18:

An example of a covariant tensor of order two is the metric tensor g_{ij}, i.e.,

$$\bar{g}_{mn} = g_{ij}\frac{\partial x^i}{\partial X^m}\frac{\partial x^j}{\partial X^n}.$$

□

Definition 2.8:

Whenever the nine quantities A^i_j in a coordinate system (x^1, x^2, x^3) are related to nine quantities $\bar{A}^m_n$ in a coordinate system $(X^1, X^2, , X^3)$ such that the transformation law

$$\bar{A}^m_n(\underset{\sim}{X}) = A^i_j(\underset{\sim}{x})\frac{\partial X^m}{\partial x^i}\frac{\partial x^j}{\partial X^n}$$

is satisfied, then these quantities are called *components of a mixed tensor of order two.* It is contravariant and covariant of order one.

□

Example 2.19:
An example of a mixed tensor of order two is the Kronecker delta symbol δ^i_j , i.e.,

$$\bar{\delta}^m_n = \delta^i_j \frac{\partial X^m}{\partial x^i} \frac{\partial x^j}{\partial X^n}.$$

□

Note:

1. For completeness, we remark that the above definitions are for *absolute* tensors. There are similar definitions for *relative* tensors, except that on the RHS of these expressions we have a coefficient J^W, e.g.,

$$\bar{\phi}(\underset{\sim}{X}) = J^W \phi(\underset{\sim}{x}) \qquad \text{(relative scalar)}$$

$$\bar{A}_m(\underset{\sim}{X}) = J^W A_i(\underset{\sim}{x}) \frac{\partial x^i}{\partial X^m} \qquad \text{(relative covariant tensor)}$$

Note:

2. For the relative tensors, J is the usual Jacobian determinant.
3. Whenever $W = 0$ these quantities are called the *components of an absolute contravariant/covariant/mixed tensor of order two.*
4. In MATH312, when speaking about tensors we are referring to *absolute* tensors, unless explicitly stated otherwise.

♪

Example 2.20:
If the 27 quantities A^i_{jk} obey the transformation law

$$\bar{A}^m_{np}(\underset{\sim}{X}) = J^2 A^i_{jk}(\underset{\sim}{x}) \frac{\partial X^m}{\partial x^i} \frac{\partial x^j}{\partial X^n} \frac{\partial x^k}{\partial X^p},$$

then this is a relative mixed order tensor of order three with weight 2. It is contravariant of order 1 and covariant of order 2.

□

Question?
Can you write down the definition of a general mixed order tensor?

✠

Answer:

□

Exercise 2.15:

Let N be the last digit of your student number and W be the first digit of your student number. Write down the transformation law for a mixed relative tensor of order N and weight W, which is contravariant of order $\lfloor N/2 \rfloor$ (where $\lfloor \cdot \rfloor$ is the floor function).

✠

Answer:

□

2.5 Coordinate transformation

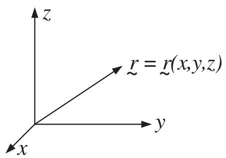
In this section, we are going to see the relationship between the metric tensor g_{ij} and the base vectors $\underset{\sim}{E}_k$.

To begin, let $(z^1, z^2, z^3) = (x, y, z)$ be the Cartesian coordinates and $(X^1, X^2, X^3) = (u, v, w)$ be a generalized curvilinear coordinate system, where the coordinates are related by

$$x = x(u, v, w), \qquad y = y(u, v, w), \qquad z = z(u, v, w),$$
$$u = u(x, y, z), \qquad v = v(x, y, z), \qquad w = w(x, y, z).$$

Consider the position vector

$$\underset{\sim}{r} = \underset{\sim}{r}(x, y, z) = x\ \underset{\sim}{e}_1 + y\ \underset{\sim}{e}_2 + z\ \underset{\sim}{e}_3,$$



where $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ are the Cartesian orthonormal base vectors. Upon rewriting the position vector in terms of u, v, and w, we find

$$\underset{\sim}{r} = x(u, v, w)\ \underset{\sim}{e}_1 + y(u, v, w)\ \underset{\sim}{e}_2 + z(u, v, w)\ \underset{\sim}{e}_3 = \underset{\sim}{r}(u, v, w),$$

so that by applying the chain rule we obtain

$$\mathrm{d}\underset{\sim}{r} = \frac{\partial \underset{\sim}{r}}{\partial u}\ \mathrm{d}u + \frac{\partial \underset{\sim}{r}}{\partial v}\ \mathrm{d}v + \frac{\partial \underset{\sim}{r}}{\partial w}\ \mathrm{d}w.$$

Expressing $\underset{\sim}{r}$ in terms of the three linearly independent unit vectors in the directions of u, v and w respectively, namely $(\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3)$, gives

$$\begin{aligned} & \underset{\sim}{r} = \underset{\sim}{r}(u, v, w) = u\underset{\sim}{E}_1 + v\underset{\sim}{E}_2 + w\underset{\sim}{E}_3, \\ \Rightarrow\quad & \mathrm{d}\underset{\sim}{r} = \mathrm{d}u\ \underset{\sim}{E}_1 + \mathrm{d}v\ \underset{\sim}{E}_2 + \mathrm{d}w\ \underset{\sim}{E}_3, \\ \text{i.e.,}\quad & \underset{\sim}{E}_1 = \frac{\partial \underset{\sim}{r}}{\partial u}, \qquad \underset{\sim}{E}_2 = \frac{\partial \underset{\sim}{r}}{\partial v}, \qquad \underset{\sim}{E}_3 = \frac{\partial \underset{\sim}{r}}{\partial w}. \end{aligned}$$

Thus, in general, the unit vectors of (u, v, w) are given by

$$\underset{\sim}{E}_i = \frac{\partial \underset{\sim}{r}}{\partial X^i}.$$

Now, we know that if $\underset{\sim}{e}_i$ and $\underset{\sim}{e}_j$ are two orthonormal base vectors, then

$$\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij},$$

and we want to see if there is a similar relationship between $\underset{\sim}{E}_i$ and $\underset{\sim}{E}_j$. Thus,

$$\begin{aligned} \underset{\sim}{E}_i \cdot \underset{\sim}{E}_j &= \frac{\partial \underset{\sim}{r}}{\partial X^i} \cdot \frac{\partial \underset{\sim}{r}}{\partial X^j}, \\ &= \left(\underset{\sim}{e}_k \frac{\partial z^k}{\partial X^i}\right) \cdot \left(\underset{\sim}{e}_m \frac{\partial z^m}{\partial X^j}\right), \qquad \underset{\sim}{r} = \underset{\sim}{r}(x, y, z) \\ &= \frac{\partial z^k}{\partial X^i}\frac{\partial z^m}{\partial X^j}(\underset{\sim}{e}_k \cdot \underset{\sim}{e}_m), \end{aligned}$$

$$\begin{aligned} &= \frac{\partial z^k}{\partial X^i}\frac{\partial z^m}{\partial X^j}\delta_{km}, \\ &= \frac{\partial z^k}{\partial X^i}\frac{\partial z^k}{\partial X^j}, \\ &= g_{ij}. \end{aligned}$$

Therefore,

$$g_{ij} = \underset{\sim}{E}_i \cdot \underset{\sim}{E}_j.$$

Example 2.21:

The transformation equations from rectangular Cartesian coordinates to spherical cylindrical coordinates can be expressed as

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\theta,$$

where here $(X^1, X^2, X^3) = (r, \theta, \phi)$. In this case, the position vector

$\underset{\sim}{r}$ can be expressed as

$$\underset{\sim}{r} = \underset{\sim}{r}(r,\theta,\phi) = r\sin\theta\cos\phi\ \underset{\sim}{e}_1 + r\sin\theta\sin\phi\ \underset{\sim}{e}_2 + r\cos\theta\ \underset{\sim}{e}_3,$$

so that

$$\frac{\partial \underset{\sim}{r}}{\partial X^1} = \frac{\partial \underset{\sim}{r}}{\partial r} = \sin\theta\cos\phi\ \underset{\sim}{e}_1 + \sin\theta\sin\phi\ \underset{\sim}{e}_2 + \cos\theta\ \underset{\sim}{e}_3 = \underset{\sim}{E}_1,$$
$$\frac{\partial \underset{\sim}{r}}{\partial X^2} = \frac{\partial \underset{\sim}{r}}{\partial \theta} = r\cos\theta\cos\phi\ \underset{\sim}{e}_1 + r\cos\theta\sin\phi\ \underset{\sim}{e}_2 - r\sin\theta\ \underset{\sim}{e}_3 = \underset{\sim}{E}_2,$$
$$\frac{\partial \underset{\sim}{r}}{\partial X^3} = \frac{\partial \underset{\sim}{r}}{\partial \phi} = -r\sin\theta\sin\phi\ \underset{\sim}{e}_1 + r\sin\theta\cos\phi\ \underset{\sim}{e}_2 = \underset{\sim}{E}_3.$$

Now, using $g_{ij} = \underset{\sim}{E}_i \cdot \underset{\sim}{E}_j$, we find

$$\begin{aligned} g_{11} &= \underset{\sim}{E}_1 \cdot \underset{\sim}{E}_1, \\ &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\cdot(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \\ &= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1. \end{aligned}$$

Similarly,

$$\begin{aligned} g_{12} &= \underset{\sim}{E}_1 \cdot \underset{\sim}{E}_2, \\ &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\cdot(r\cos\theta\cos\phi, r\cos\theta\sin\phi, -r\sin\theta), \\ &= r\sin\theta\cos\theta\cos^2\phi + r\sin\theta\cos\theta\sin^2\phi - r\sin\theta\cos\theta = 0. \end{aligned}$$

If you continue to calculate all the other values, you will obtain

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

□

Exercise 2.16:

Consider the coordinate transformation from Cartesian coordinates into cylindrical polar coordinates, namely

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

where $(X^1, X^2, X^3) = (r,\theta,z)$. Find the unit vectors $\underset{\sim}{E}_i$ in the direction X^i, and then use the relationship $\underset{\sim}{E}_i \cdot \underset{\sim}{E}_j = g_{ij}$ to find the metric tensor.

✠

Answer:

□

Maple Exercise 2.1:

Use Maple to create a function that given a position vector $\underset{\sim}{r} = \underset{\sim}{r}(X^1, X^2, X^3)$ then produces the metric tensor and the conjugate metric tensor. (HINT: you may find it useful to consider using the `module` command)

(All Maple exercises given in the lecture notes are optional)

✠

2.6 A use for metric tensors

2.6.1 Raising and lowering indices

One use of the metric and conjugate metric tensors is to *raise* and *lower* the indices of higher-order tensors. For example, to raise an index of A_{mn}, we use the conjugate metric tensor, such that

$$A^i_n = g^{mi} A_{mn},$$

or

$$A^i_m = g^{in} A_{mn}.$$

Note:

The two mixed tensors A^i_n and A^i_m are not necessarily the same.

♪

Both indices are raised by repeating the process, i.e.,

$$A^{ij} = g^{mi} g^{nj} A_{mn}.$$

To lower an index, we use a similar approach, but instead we use the metric tensor, for example,

$$\begin{aligned} A_i^n &= g_{mi} A^{mn}, \\ A_i^m &= g_{in} A^{mn}, \\ A_{ij} &= g_{mi} g_{nj} A^{mn}. \end{aligned}$$

Note:

1. The tensors obtained by raising or lowering indices are called *associated tensors.*
2. $g_{ij} = g_{ji}$ and $g^{jk} = g^{kj} \quad \Rightarrow \quad g_{mi} A^{mn} = g_{im} A^{mn}$, etc.

♪

Exercise 2.17:

For the following tensors, lower all superscripts and raise all subscripts.

1. A^k, **Answer:**
2. A^i_j,
3. δ_{ij},
4. E^{mn}_{ijk},
5. g_{ij},

□

Note:

In 5., we can also use the identity $g^{nj} g_{ij} = \delta^n_i$ to prove the result.

♪

Summary

In this lecture, we ...

- considered covariant/contravariant tensors of order n
- examined how to express the metric tensor in terms of base vectors

Coming up

In the next lecture, we ...

- consider the physical components of tensors
- introduce direction cosines

Homework Exercise 2.6:

1. Consider the transformation equations for Cartesian coordinates into parabolic cylindrical coordinates, namely

$$x = \xi\eta, \qquad y = \frac{1}{2}(\xi^2 - \eta^2), \qquad z = z$$

where $(X^1, X^2, X^3) = (\xi, \eta, z)$. Find the unit vectors $\underset{\sim}{E}_i$ in the direction X^i, and then use the relationship $\underset{\sim}{E}_i \cdot \underset{\sim}{E}_j = g_{ij}$ to find the metric tensor.

2. Let A_{ij} be a tensor in cylindrical polar coordinates, given by

$$A_{ij} = \begin{pmatrix} 1 & r & 0 \\ r & -r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the three associated tensors A_i^m, A_j^n and A^{mn}.