## Review

In the previous lecture, we ...

- introduced the $\varepsilon - \delta$ identity
- introduced the concepts of metric tensors

## Aims

In this lecture, we will ...

- introduce the conjugate metric tensor
- introduce the concepts of contravariant and covariant tensors

**Example 2.11:**

If the metric tensor $g_{ij}$ is given by

$$g_{ij} = \frac{\partial z^k}{\partial x^i}\frac{\partial z^k}{\partial x^j},$$

then calculate the metric tensor for parabolic cylindrical coordinates $(\xi, \eta, z)$, where

$$x = \xi\eta, \quad y = \frac{1}{2}(\xi^2 - \eta^2), \quad z = z.$$

✠

**Answer:**

$$g_{11} = \frac{\partial z^k}{\partial x^1}\frac{\partial z^k}{\partial x^1} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 = \eta^2 + \xi^2.$$

$$g_{12} = \frac{\partial z^k}{\partial x^1}\frac{\partial z^k}{\partial x^2} = \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} = \eta\xi - \xi\eta + 0 = 0.$$

and so forth. Finally, we obtain

$$g_{ij} = \begin{pmatrix} \eta^2 + \xi^2 & 0 & 0 \\ 0 & \eta^2 + \xi^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

**Exercise 2.12:**

Consider the modified cylindrical polar coordinates $(r, \theta, z)$ where

$$x = r\cos\theta + r, \quad y = r\sin\theta, \quad z = z.$$

Find the metric tensor.

✠

**Answer:**

□

**Note:**
In this exercise, not all of the off-diagonal elements are zero, i.e., $g_{12} = g_{21} = -r\sin\theta$. But even so, the metric tensor is *still* symmetric about the diagonal.

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### 2.3.3 Conjugate metric tensor

For reasons seen a little bit later, sometimes it is more convenient to use the inverse of the metric tensor. This inverse is called the *conjugate metric tensor*, and is denoted by $g^{jk}$.

As $g^{jk}$ is the inverse of $g_{ij}$ then

$$g_{ij}\ g^{jk} = \delta_i^k.$$

To calculate $g^{jk}$, the usual method is to use *cofactors*, namely

$$g^{jk} = \frac{\text{cofactor}(g_{jk})}{g},$$

where $g$ is the determinant of $g_{ij}$, i.e.,

$$g = \left|\frac{\partial z^k}{\partial x^i}\frac{\partial z^k}{\partial x^j}\right| = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}.$$

**Question?**

What is a cofactor?

✠

**Answer:** (From MATH187)

□

**Example 2.12:**
Previously, for the modified cylindrical polar coordinates $(r, \theta, z)$, where

$$x = r\cos\theta + r, \quad y = r\sin\theta, \quad z = z,$$

we found the metric tensor

$$g_{ij} = \begin{pmatrix} 2\cos\theta + 2 & -r\sin\theta & 0 \\ -r\sin\theta & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To find the conjugate metric tensor $g^{jk}$, we first need the determinant $g$, namely

$$g = \begin{vmatrix} 2\cos\theta + 2 & -r\sin\theta & 0 \\ -r\sin\theta & r^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = r^2(2\cos\theta - \sin^2\theta + 2) = r^2(\cos\theta + 1)^2,$$

so that from the formulae

$$g^{jk} = \frac{(-1)^{j+k} \times G_{jk}}{g},$$

we find

$$g^{11} = \frac{(-1)^{1+1}\begin{vmatrix} r^2 & 0 \\ 0 & 1 \end{vmatrix}}{r^2(\cos\theta + 1)^2} = \frac{1}{(\cos\theta + 1)^2},$$

$$g^{12} = \frac{(-1)^{1+2}\begin{vmatrix} -r\sin\theta & 0 \\ 0 & 1 \end{vmatrix}}{r^2(\cos\theta + 1)^2} = \frac{\sin\theta}{r(\cos\theta + 1)^2},$$

etc.

Finally, we obtain

$$g^{jk} = \begin{pmatrix} \frac{1}{(\cos\theta+1)^2} & \frac{\sin\theta}{r(\cos\theta+1)^2} & 0 \\ \frac{\sin\theta}{r(\cos\theta+1)^2} & \frac{2}{r^2(\cos\theta+1)} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

**Exercise 2.13:**

In Exercise 2.10, we found the metric tensor for cylindrical polar coordinates is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the conjugate metric tensor. ✠

**Answer:**

□

**Question?**

In Exercise 2.13, there is an easier way to calculate $g^{jk}$. What is it? ✠

**Answer:**

□

**Note:**

A problem would occur if a diagonal element of $g_{ij}$ was zero. However, from the definition of $g_{ij}$, this would only happen if the coordinate transformation

$$g_{\underline{i}\,\underline{i}} = (z^1{}_{,\underline{i}})^2 + (z^2{}_{,\underline{i}})^2 + (z^3{}_{,\underline{i}})^2 = 0 \qquad \Rightarrow \qquad z^k = z^k(x^1, x^2, x^3) = 0.$$

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### 2.3.4 Relationship between $g$ and $J$

From section 2.3, recall that

$$J = \left|\frac{\partial X^i}{\partial x^j}\right| = \begin{vmatrix} \dfrac{\partial X^1}{\partial x^1} & \dfrac{\partial X^1}{\partial x^2} & \dfrac{\partial X^1}{\partial x^3} \\ \dfrac{\partial X^2}{\partial x^1} & \dfrac{\partial X^2}{\partial x^2} & \dfrac{\partial X^2}{\partial x^3} \\ \dfrac{\partial X^3}{\partial x^1} & \dfrac{\partial X^3}{\partial x^2} & \dfrac{\partial X^3}{\partial x^3} \end{vmatrix},$$

and if $A$ and $B$ are matrices, then $|AB| = |A||B|$ and $|A^T| = |A|$.

Thus, if we assume

$$X^i = z^i(x^1, x^2, x^3),$$

then from $J^2 = |J_{ij}| \times |J_{ij}| = |J_{ij}^T| \times |J_{ij}|$, we find ...

$$J^2 = \left| \begin{pmatrix} \dfrac{\partial z^1}{\partial x^1} & \dfrac{\partial z^2}{\partial x^1} & \dfrac{\partial z^3}{\partial x^1} \\ \dfrac{\partial z^1}{\partial x^2} & \dfrac{\partial z^2}{\partial x^2} & \dfrac{\partial z^3}{\partial x^2} \\ \dfrac{\partial z^1}{\partial x^3} & \dfrac{\partial z^2}{\partial x^3} & \dfrac{\partial z^3}{\partial x^3} \end{pmatrix} \begin{pmatrix} \dfrac{\partial z^1}{\partial x^1} & \dfrac{\partial z^1}{\partial x^2} & \dfrac{\partial z^1}{\partial x^3} \\ \dfrac{\partial z^2}{\partial x^1} & \dfrac{\partial z^2}{\partial x^2} & \dfrac{\partial z^2}{\partial x^3} \\ \dfrac{\partial z^3}{\partial x^1} & \dfrac{\partial z^3}{\partial x^2} & \dfrac{\partial z^3}{\partial x^3} \end{pmatrix} \right|,$$

$$= \begin{vmatrix} \dfrac{\partial z^k}{\partial x^1}\dfrac{\partial z^k}{\partial x^1} & \dfrac{\partial z^k}{\partial x^1}\dfrac{\partial z^k}{\partial x^2} & \dfrac{\partial z^k}{\partial x^1}\dfrac{\partial z^k}{\partial x^3} \\ \dfrac{\partial z^k}{\partial x^2}\dfrac{\partial z^k}{\partial x^1} & \dfrac{\partial z^k}{\partial x^2}\dfrac{\partial z^k}{\partial x^2} & \dfrac{\partial z^k}{\partial x^2}\dfrac{\partial z^k}{\partial x^3} \\ \dfrac{\partial z^k}{\partial x^3}\dfrac{\partial z^k}{\partial x^1} & \dfrac{\partial z^k}{\partial x^3}\dfrac{\partial z^k}{\partial x^2} & \dfrac{\partial z^k}{\partial x^3}\dfrac{\partial z^k}{\partial x^3} \end{vmatrix},$$

$$= g.$$

Thus: $\quad g = J^2$ or $J = \sqrt{g}$.

**Note:**

If $J = 0$, it means that there is no unique inverse, so that the metric tensor $g_{ij}$ will have a zero determinant, i.e., $g = 0$.

If $J$ is known, then it is easy to find $g$, and vice-versa.

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**Example 2.13:**

In Example 2.13, we considered the coordinate transformation

$$x = r\cos\theta + r, \qquad y = r\sin\theta, \qquad z = z,$$

where we found

$$g = \left|\frac{\mathrm{d}z^k}{\mathrm{d}x^i}\frac{\mathrm{d}z^k}{\mathrm{d}x^j}\right| = r^2(\cos\theta + 1)^2.$$

To check, find $J$, i.e.,

$$J = \left|\frac{\partial z^i}{\partial x^j}\right| = \begin{vmatrix} \cos\theta + 1 & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos\theta(\cos\theta + 1) + r\sin^2\theta$$

$$= r(\cos\theta + 1).$$

Thus,

$$J^2 = [r(\cos\theta + 1)]^2 = r^2(\cos\theta + 1)^2 = g.$$

□

**Note:**

We will see later that we need $J$ in various equations, where we have already worked out $g_{ij}$. In such situations, it is much easier to calculate $J$ using $J = \sqrt{g}$, than it is using $J = \left|\frac{\mathrm{d}z^i}{\mathrm{d}x^j}\right|$.

♪

## 2.4 Classifying tensors

We now look at how to further classify tensors. To do this, we introduce the following definitions:

**Definition 2.3:**

A function $\phi(x^1, x^2, x^3)$ is called a *scalar* if upon transformation of coordinates $x^k = x^k(X^1, X^2, X^3)$ it does not change its original value, i.e.,

$$\phi(x^1(\underset{\sim}{X}), x^2(\underset{\sim}{X}), x^3(\underset{\sim}{X})) \equiv \Phi(X^1, X^2, X^3) = \phi(x^1, x^2, x^3).$$

$\square$

**Example 2.14:**

An example of a scalar is the temperature at a point, for is has the same value in all coordinate systems.

$\square$

**Definition 2.4:**

Quantities $A^k(\underset{\sim}{x})$ are called *contravariant components of a vector* (or simply a *contravariant vector*) if under coordinate transformation $x^k = x^k(X^1, X^2, X^3)$ they transform according to the rule

$$\bar{A}^i(\underset{\sim}{X}) = A^j(\underset{\sim}{x})\frac{\partial X^i}{\partial x^j}.$$

Here, $\bar{A}^i$ are the components of the same vector, but with reference to coordinates $X^j$.

$\square$

**Example 2.15:**

An example of a contravariant vector is the differential, i.e., $A^i = \mathrm{d}x^i$, since according to the chain rule we have

$$\mathrm{d}X^i = \frac{\partial X^i}{\partial x^j}\mathrm{d}x^j.$$

$\square$

**Definition 2.5:**

Similarly, $A_k(\underset{\sim}{x})$ are called *covariant components of a vector* (or simply a *covariant vector*) if under coordinate transformation $x^k = x^k(X^1, X^2, X^3)$ they transform according to the rule

$$\bar{A}_i(\underset{\sim}{X}) = A_j(\underset{\sim}{x})\frac{\partial x^j}{\partial X^i}.$$

$\square$

**Example 2.16:**

An example of a covariant vector is the partial derivative of a scalar, i.e., $A_i = \frac{\partial \phi}{\partial x^i}$, since

$$\frac{\partial \phi}{\partial X^j} = \frac{\partial \phi}{\partial x^i}\frac{\partial x^i}{\partial X^j}.$$

$\square$

**Exercise 2.14:**

For the following tensors, identify whether scalar, covariant or contravariant and write the transform transformations:

1. $A^i$ **Answer:**
2. $A$
3. $A_i$
4. $g_{ij}$
5. $g^{jk}$
6. $\delta^i_j$

$\square$

# Summary

In this lecture, we . . .

- introduced the conjugate metric tensor
- introduced the concepts of contravariant and covariant tensors

# Coming up

In the next lecture, we . . .

- consider covariant/contravariant tensors of order $n$
- examine how to express the metric tensor in terms of base vectors

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**Homework Exercise 2.5:**

1. For the spherical polar coordinate system $(r, \theta, \phi)$, which is related to the Cartesian coordinate system by

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\theta,$$

   find the conjugate metric tensor.

2. Give an example, with appropriate transformation rules, of the following tensors:
   (a) A scalar.
   (b) A contravariant tensor.
   (c) A covariant tensor.
   (d) A mixed tensor.