Review

In the previous lecture, we …

- introduced the permutation symbol
- expressed some common vector operations in terms of tensors

Aims

In this lecture, we will …

- introduce the $\varepsilon - \delta$ identity
- introduce the concepts of metric tensors

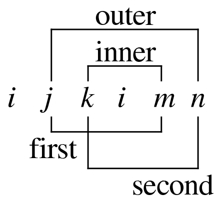
2.2.6 The $\varepsilon - \delta$ identity

The Kronecker delta and the permutation symbol are very important quantities that will appear again and again in this subject. They are connected by the identity

$$\varepsilon_{ijk}\,\varepsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}.$$

This identity is used frequently. It can be proven by using vector products.

A useful tool to remember the order is shown graphically.



Example 2.8:

Show the vector identity $\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{a}) = \underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{a}) - \nabla^2 \underset{\sim}{a}$.

✠

Answer:

$$\begin{aligned}
\text{LHS} &= \left(\underset{\sim}{e}_m \frac{\partial}{\partial x^m}\right) \times \left[\left(\underset{\sim}{e}_i \frac{\partial}{\partial x^i}\right) \times (a_j \underset{\sim}{e}_j)\right], \\
&= \left(\underset{\sim}{e}_m \frac{\partial}{\partial x^m}\right) \times \varepsilon_{ijk} a_{j,i} \underset{\sim}{e}_k, \\
&= \varepsilon_{ijk}\varepsilon_{mkn} a_{j,im} \underset{\sim}{e}_n, \\
&= \varepsilon_{kij}\varepsilon_{knm} a_{j,im} \underset{\sim}{e}_n, \\
&= (\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn}) a_{j,im} \underset{\sim}{e}_n, \\
&= a_{j,ij} \underset{\sim}{e}_i - a_{j,ii} \underset{\sim}{e}_j.
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \left(\underset{\sim}{e}_m \frac{\partial}{\partial x^m}\right) \left[\left(\underset{\sim}{e}_i \frac{\partial}{\partial x^i}\right) \cdot (a_j \underset{\sim}{e}_j)\right] - \left[\left(\underset{\sim}{e}_s \frac{\partial}{\partial x^s}\right) \cdot \left(\underset{\sim}{e}_t \frac{\partial}{\partial x^t}\right)\right] a_r \underset{\sim}{e}_r, \\
&= \underset{\sim}{e}_m \frac{\partial}{\partial x^m} \left[a_{j,i}\delta_{ij}\right] - \left[\frac{\partial^2}{\partial x^s \partial x^t}\delta_{st}\right] a_r \underset{\sim}{e}_r, \\
&= a_{i,im} \underset{\sim}{e}_m - a_{r,ss} \underset{\sim}{e}_r, \\
&= \text{LHS}.
\end{aligned}$$

□

Note:

In the example above, the names of indices were different. However, as they are dummy indices, then the actual name is irrelevant.

Also, we have implicitly assumed

$$a_{i,ij} = a_{i,ji}.$$

♪

Exercise 2.9:

Prove the vector identity

$\underset{\sim}{\nabla} \times (\underset{\sim}{A} \times \underset{\sim}{B}) = \underset{\sim}{A}(\underset{\sim}{\nabla} \cdot \underset{\sim}{B}) - \underset{\sim}{B}(\underset{\sim}{\nabla} \cdot \underset{\sim}{A}) + (\underset{\sim}{B} \cdot \underset{\sim}{\nabla})\underset{\sim}{A} - (\underset{\sim}{A} \cdot \underset{\sim}{\nabla})\underset{\sim}{B}$. ✠

Answer:

$\square$

2.3 Curvilinear coordinates

Consider two coordinate systems in $\mathbb{R}^3$, say (x^1, x^2, x^3) and (X^1, X^2, X^3). Assuming the three functions

$$x^k = x^k(X^1, X^2, X^3),$$

represent a point P, then if there exists a *unique* inverse

$$X^k = X^k(x^1, x^2, x^3),$$

the coordinates (x^1, x^2, x^3) and (X^1, X^2, X^3) are *curvilinear*.

It can be shown that a unique inverse exists if the Jacobian determinant is nonzero, i.e.,

$$J = \left|\frac{\partial x^i}{\partial X^j}\right| = \begin{vmatrix} \dfrac{\partial x^1}{\partial X^1} & \dfrac{\partial x^1}{\partial X^2} & \dfrac{\partial x^1}{\partial X^3} \\ \dfrac{\partial x^2}{\partial X^1} & \dfrac{\partial x^2}{\partial X^2} & \dfrac{\partial x^2}{\partial X^3} \\ \dfrac{\partial x^3}{\partial X^1} & \dfrac{\partial x^3}{\partial X^2} & \dfrac{\partial x^3}{\partial X^3} \end{vmatrix} \neq 0.$$

Note:

It is implicity assumed that the transformation equations are real and continuous, and that all derivatives exist and are also real and continuous.

The inverse transformation equations are usually found from geometrical considerations. ♪

Example 2.9:

Consider the transformation from cylindrical polar coordinates (X^1, X^2, X^3) to rectangular Cartesian coordinates (x^1, x^2, x^3):

$$x^1 = X^1 \cos X^2, \qquad x^2 = X^1 \sin X^2, \qquad x^3 = X^3.$$

Then the Jacobian determinant becomes

$$J = \begin{vmatrix} \cos X^2 & -X^1 \sin X^2 & 0 \\ \sin X^2 & X^1 \cos X^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = X^1,$$

so there exists a unique inverse everywhere except along $X^1 = 0$. In particular,

$$X^1 = \sqrt{(x^1)^2 + (x^2)^2}, \qquad X^2 = \arctan \frac{x^2}{x^1}, \qquad X^3 = x^3.$$

$\square$

2.3.1 Calculation of derivatives

Given the curvilinear coordinates defined previously, we now consider the differentiation of a scalar function $\Phi(X^1, X^2, X^3)$ with respect to x^i, which using the chain rule gives

$$\frac{\partial \Phi}{\partial x^i} = \frac{\partial \Phi}{\partial X^1}\frac{\partial X^1}{\partial x^i} + \frac{\partial \Phi}{\partial X^2}\frac{\partial X^2}{\partial x^i} + \frac{\partial \Phi}{\partial X^3}\frac{\partial X^3}{\partial x^i} = \frac{\partial \Phi}{\partial X^j}\frac{\partial X^j}{\partial x^i}.$$

Further, the second partial derivative of Φ with respect to x^m is

$$\begin{aligned}\frac{\partial^2 \Phi}{\partial x^i \partial x^m} &= \frac{\partial \Phi}{\partial X^j}\frac{\partial^2 X^j}{\partial x^i \partial x^m} + \frac{\partial}{\partial x^m}\left\{\frac{\partial \Phi}{\partial X^j}\right\}\frac{\partial X^j}{\partial x^i},\\ &= \frac{\partial \Phi}{\partial X^j}\frac{\partial^2 X^j}{\partial x^i \partial x^m} + \frac{\partial^2 \Phi}{\partial X^j \partial X^k}\frac{\partial X^k}{\partial x^m}\frac{\partial X^j}{\partial x^i},\end{aligned}$$

noting that k and j are dummy indices, and hence summed over.

Exercise 2.10:

Let $\Phi = \Phi(r, \theta)$, where (r, θ) are cylindrical polar coordinates that are related to the Cartesian coordinates (x, y) by the transformation equations

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Find the partial derivatives $\dfrac{\partial \Phi}{\partial x}$ and $\dfrac{\partial^2 \Phi}{\partial x^2}$.

✠

Answer:

□

We have just looked at how to calculate the derivatives between any two curvilinear coordinates.

A question that should jump to mind is: Why?

The answer is that while we want to use tensors to express equations in coordinate free notation, we also want to be able to express the tensor equations in terms of a particular coordinate system in order to solve a problem in the real world.

And, as differential equations are abundant in the real world, then we need to know how to express the derivative of one coordinate system with respect to another.

2.3.2 The metric tensor g_{ij}

If we only consider problems in Cartesian coordinates, then calculating derivatives is easy, as illustrated in Lecture 3 when considering vector operations with $\underset{\sim}{\nabla}$.

However, we saw in Exercise 2.9 that when calculating the derivatives of other curvilinear coordinates, then life gets more complicated.

To help, let us introduce the *metric tensor* g_{ij}, which is defined by

$$g_{ij} = \frac{\partial z^k}{\partial x^i}\frac{\partial z^k}{\partial x^j},$$

where z^k denotes Cartesian coordinates.

For some background, consider the distance squared between two points z^i and $z^i + \mathrm{d}z^i$, namely

$$(\mathrm{d}s)^2 = (\mathrm{d}z^1)^2 + (\mathrm{d}z^2)^2 + (\mathrm{d}z^3)^2 = \mathrm{d}z^k\mathrm{d}z^k.$$

Now, if $z^i = z^i(x^1, x^2, x^3)$, then

$$\mathrm{d}z^i = \frac{\partial z^i}{\partial x^j}\mathrm{d}x^j,$$

so that

$$(\mathrm{d}s)^2 = \frac{\partial z^k}{\partial x^i}\frac{\partial z^k}{\partial x^j}\mathrm{d}x^i\mathrm{d}x^j = g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j.$$

Therefore, g_{ij} is called a *metric* of the spaced defined by the coordinates (x^1, x^2, x^3).

Note:

The metric tensor is symmetric, i.e., $g_{ij} = g_{ji}$.

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The element of arc length is invariant in terms of coordinate systems, so if we have two different coordinate systems (x^1, x^2, x^3) and $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, then considering the distance squared between the two points z^i and $z^i + \mathrm{d}z^i$ gives

$$(\mathrm{d}s)^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \bar{g}_{mn}\mathrm{d}\bar{x}^m\mathrm{d}\bar{x}^n.$$

Now, if we assume $x^i = x^i(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, then

$$\mathrm{d}x^i = \frac{\partial x^i}{\partial \bar{x}^m}\mathrm{d}\bar{x}^m,$$

so that

$$g_{ij}\frac{\partial x^i}{\partial \bar{x}^m}\frac{\partial x^j}{\partial \bar{x}^n}\mathrm{d}\bar{x}^m\bar{x}^n = \bar{g}_{mn}\mathrm{d}\bar{x}^m\mathrm{d}\bar{x}^n,$$

i.e.,

$$\bar{g}_{mn} = g_{ij}\frac{\partial x^i}{\partial \bar{x}^m}\frac{\partial x^j}{\partial \bar{x}^n}.$$

Therefore, g_{ij} transforms as a second order covariant tensor (we will look at what this means a little later).

Example 2.10:

If $(x^1, x^2, x^3) = (x, y, z)$, then

$$g_{ij} = \frac{\partial z^k}{\partial x^i}\frac{\partial z^k}{\partial x^j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

Exercise 2.11:

Let (x^1, x^2, x^3) be the usual cylindrical polar coordinates (r, θ, z). Thus,

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Determine the metric tensor g_{ij}.

✠

Answer:

□

Note:

As required, the metric tensors calculated above are symmetric.

Although the off diagonal terms in the above examples of the metric tensor are all zero, this is not always true.

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Summary

In this lecture, we ...

- introduced the $\varepsilon - \delta$ identity
- introduced the concepts of metric tensors

Coming up

In the next lecture, we ...

- introduce the conjugate metric tensor
- introduce the concepts of contravariant and covariant tensors

Homework Exercise 2.4:

1. Using index notation prove:
   $(\underset{\sim}{a} \times \underset{\sim}{b}) \times (\underset{\sim}{c} \times \underset{\sim}{d}) = \underset{\sim}{c}(\underset{\sim}{d} \cdot \underset{\sim}{a} \times \underset{\sim}{b}) - \underset{\sim}{d}(\underset{\sim}{c} \cdot \underset{\sim}{a} \times \underset{\sim}{b})$.
   You will need to use the $\varepsilon - \delta$ identity.

2. For the spherical polar coordinate system (r, θ, ϕ), which is related to the Cartesian coordinate system by
   $$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\theta,$$
   find the metric tensor.