Review

In the previous lecture, we ...

- considered how to classify tensors and basic operations
- introduced the Kronecker delta symbol

Aims

In this lecture, we will ...

- introduce the permutation symbol
- express some common vector operations in terms of tensors

Example 2.3:

In Cartesian coordinates, the usual set of orthonormal base vectors is $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} = \{\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}\}$. Show $\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$.

✠

Answer:

$$\underset{\sim}{e}_1 = (1,0,0), \quad \underset{\sim}{e}_2 = (0,1,0), \quad \underset{\sim}{e}_3 = (0,0,1), \quad |\underset{\sim}{e}_i| = 1, i = 1,2,3.$$

Thus, the following dot products are obtained:

$$\begin{aligned} \underset{\sim}{e}_1 \cdot \underset{\sim}{e}_1 = 1, \quad & \underset{\sim}{e}_1 \cdot \underset{\sim}{e}_2 = 0, \quad & \underset{\sim}{e}_1 \cdot \underset{\sim}{e}_3 = 0, \\ \underset{\sim}{e}_2 \cdot \underset{\sim}{e}_1 = 0, \quad & \underset{\sim}{e}_2 \cdot \underset{\sim}{e}_2 = 1, \quad & \underset{\sim}{e}_2 \cdot \underset{\sim}{e}_3 = 0, \\ \underset{\sim}{e}_3 \cdot \underset{\sim}{e}_1 = 0, \quad & \underset{\sim}{e}_3 \cdot \underset{\sim}{e}_2 = 0, \quad & \underset{\sim}{e}_3 \cdot \underset{\sim}{e}_3 = 1. \end{aligned}$$

Therefore, if $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\} = \{\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}\}$, then $\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$. □

Exercise 2.5:

Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ be any set of orthonormal base vectors. Show

$$\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}.$$

✠

Answer:

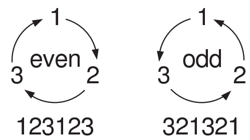
□

The second special tensor we are interested in is the *permutation symbol*, ε_{ijk}.

Definition 2.2:

The permutation symbol, ε_{ijk}, is a tensor of order 3 and is defined by

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if even permutation,} \\ -1, & \text{if odd permuation,} \\ 0, & \text{otherwise.} \end{cases}$$



□

Even permutation: $(i,j,k) = (1,2,3), (3,1,2)$ or $(2,3,1)$.
Odd permutation: $(i,j,k) = (3,2,1), (2,1,3)$ or $(1,3,2)$.

Also,

$$\varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i).$$

Example 2.4:

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \qquad \varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1,$$
$$\varepsilon_{112} = \varepsilon_{111} = \cdots = 0.$$

□

Example 2.5:

Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ be a set of orthonormal base vectors, where $\underset{\sim}{e}_i = (e_{i1}, e_{i2}, e_{i3})$. Then the cross-product of $\underset{\sim}{e}_i \times \underset{\sim}{e}_j$ is given by

$$\begin{aligned}\underset{\sim}{e}_i \times \underset{\sim}{e}_j &= \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ e_{i1} & e_{i2} & e_{i3} \\ e_{j1} & e_{j2} & e_{j3} \end{vmatrix}, \\ &= \underset{\sim}{e}_1(e_{i2}e_{j3} - e_{j2}e_{i3}) - \underset{\sim}{e}_2(e_{i1}e_{j3} - e_{j1}e_{i3}) + \underset{\sim}{e}_3(e_{i1}e_{j2} - e_{j1}e_{i2}), \\ &= \varepsilon_{ijk}\underset{\sim}{e}_k.\end{aligned}$$

□

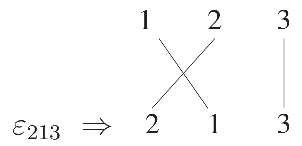
When looking at vectors in the following sections, it is important to remember the following two identities:

$$\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij},$$
$$\underset{\sim}{e}_i \times \underset{\sim}{e}_j = \varepsilon_{ijk}\underset{\sim}{e}_k,$$

where $\underset{\sim}{e}_i$ are orthonormal base vectors.

Note:

An alternate way to evaluate the permutation system is shown graphically below:



Join numbers, and then count number of lines that cross to see if even or odd.

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Exercise 2.6:

Evaluate the following expressions:

1. $3\varepsilon_{123} + 2\varepsilon_{213} - \varepsilon_{122}$.
2. $\varepsilon_{312}\varepsilon_{321} + \varepsilon_{231}\varepsilon_{131}$.
3. ε_{561234}.
4. ε_{612453}.

✠

Answer:

□

2.2.4 Vector Operations

Given what we now know about orthonormal base vectors, we now look at how we can express common vector operations in tensor notation.

This will enable us to express vector equations in tensorial form which are valid in any coordinate system.

To begin, let $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ be any three vectors and $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ be a set of orthonormal base vectors in $\mathbb{R}^3$. Thus, we can write

$$\underset{\sim}{u} = u_i\underset{\sim}{e}_i, \qquad \underset{\sim}{v} = v_j\underset{\sim}{e}_j, \qquad \underset{\sim}{w} = w_k\underset{\sim}{e}_k.$$

Operation 1: *Dot (scalar) product of two vectors*

$$\begin{aligned}
\underset{\sim}{u} \cdot \underset{\sim}{v} &= (u_i\, \underset{\sim}{e}_i) \cdot (v_j\, \underset{\sim}{e}_j), && \text{by definition} \\
&= u_i v_j (\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j), && u_i, v_j \text{ scalars} \\
&= u_i v_j \delta_{ij}, && \text{by definition} \\
&= u_i v_i. && \text{(scalar)}
\end{aligned}$$

Operation 2: *Cross (vector) product of two vectors*

$$\begin{aligned}
\underset{\sim}{u} \times \underset{\sim}{v} &= (u_i\, \underset{\sim}{e}_i) \times (v_j\, \underset{\sim}{e}_j), && \text{by definition} \\
&= u_i v_j (\underset{\sim}{e}_i \times \underset{\sim}{e}_j), && u_i, v_j \text{ scalars} \\
&= u_i v_j\ \varepsilon_{ijk}\, \underset{\sim}{e}_k, && \text{by definition} \\
&= \varepsilon_{ijk}\, u_i v_j\, \underset{\sim}{e}_k. && \text{(vector)}
\end{aligned}$$

Operation 3: *Scalar triple product of three vectors*

$$\begin{aligned}
(\underset{\sim}{u} \times \underset{\sim}{v}) \cdot \underset{\sim}{w} &= (\varepsilon_{ijk}\, u_i v_j\, \underset{\sim}{e}_k) \cdot w_m\, \underset{\sim}{e}_m, && \text{by definition} \\
&= \varepsilon_{ijk}\, u_i v_j w_m (\underset{\sim}{e}_k \cdot \underset{\sim}{e}_m), && u_i, v_j, w_m \text{ scalars} \\
&= \varepsilon_{ijk}\, u_i v_j w_m\, \delta_{km}, && \text{by definition} \\
&= \varepsilon_{ijk}\, u_i v_j w_k. && \text{(scalar)}
\end{aligned}$$

Note:

From the above definition of the scalar triple product it is trivial to prove

$$(\underset{\sim}{u} \times \underset{\sim}{v}) \cdot \underset{\sim}{w} = (\underset{\sim}{w} \times \underset{\sim}{u}) \cdot \underset{\sim}{v} = (\underset{\sim}{v} \times \underset{\sim}{w}) \cdot \underset{\sim}{u},$$

using the property of the permutation symbol: $\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$.

Operation 4: *Determinants*

The determinant of a 3×3 matrix A, namely

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

can be expressed as $|A| = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$.

(I leave you to check this is true!)

Example 2.6:

Show that $\underset{\sim}{u} \times \underset{\sim}{v} = -\underset{\sim}{v} \times \underset{\sim}{u}$.

Answer:

Let $\underset{\sim}{u} = u_i \underset{\sim}{e}_i$ and $\underset{\sim}{v} = v_j \underset{\sim}{e}_j$. Then

$$\begin{aligned}
\underset{\sim}{u} \times \underset{\sim}{v} &= \varepsilon_{ijk} u_i v_j\, \underset{\sim}{e}_k, && \text{by definition} \\
&= -\varepsilon_{jik} v_j u_i\, \underset{\sim}{e}_k, && \text{as } \varepsilon_{ijk} = -\varepsilon_{jik} \\
&= -\underset{\sim}{v} \times \underset{\sim}{u}. && \text{by definition}
\end{aligned}$$

Thus, $\underset{\sim}{u} \times \underset{\sim}{v} = -\underset{\sim}{v} \times \underset{\sim}{u}$.

□

Exercise 2.7:

If $\underset{\sim}{a} = a_i \underset{\sim}{e}_i$, $\underset{\sim}{b} = b_j \underset{\sim}{e}_j$ and $\underset{\sim}{c} = c_k \underset{\sim}{e}_k$, then simplify the following using the vector definitions and base vector properties.

1. $\underset{\sim}{a} \cdot \underset{\sim}{b}$.
2. $\underset{\sim}{c} \times \underset{\sim}{a}$.
3. $(\underset{\sim}{b} \times \underset{\sim}{c}) \cdot \underset{\sim}{a}$.
4. $(\underset{\sim}{b} \times \underset{\sim}{a}) \cdot (\underset{\sim}{a} \times \underset{\sim}{c})$.
5. $[(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c}]\underset{\sim}{a} \cdot \underset{\sim}{b}$.

✠

Answer:

□

2.2.5 Vector Operator $\underset{\sim}{\nabla}$

When developing a mathematical model of the real world, the resulting theory usually involves differential equations. As such, it would be beneficial to express the usual differential operators in tensorial form.

To begin, we initially assume the Cartesian coordinate system $(x^1, x^2, x^3) = (x, y, z)$ and the set of orthonormal base vectors $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$.

Then, the del operator, $\underset{\sim}{\nabla}$, can be expressed as

$$\underset{\sim}{\nabla} = \underset{\sim}{e}_1 \frac{\partial}{\partial x^1} + \underset{\sim}{e}_2 \frac{\partial}{\partial x^2} + \underset{\sim}{e}_3 \frac{\partial}{\partial x^3} = \underset{\sim}{e}_i \frac{\partial}{\partial x^i}.$$

Later, we will see how to express $\underset{\sim}{\nabla}$ in any coordinate system.

Note:

When writing derivatives in tensorial form, it is convenient to use a shorthand notation. For example,

$$\frac{\partial v_i}{\partial x^j} = v_{i,j},$$

denotes the derivative of the ith component of $\underset{\sim}{v}$ with respect to x^j.

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Example 2.7:

1. $\dfrac{\partial A_i}{\partial x^k} = A_{i,k}$
2. $\dfrac{\partial^3 F_{un}}{\partial x_3 \partial x_1 \partial x_2} = F_{un,312}$
3. $\dfrac{\partial^2 C}{\partial x_m \partial x_i} = C_{,mi}$
4. $\dfrac{\partial^3 H_e}{\partial l \partial l \partial o} = H_{e,llo}$

□

Thus, if $f(\underset{\sim}{x})$ is a scalar function, then the *gradient* of f is given by

$$\text{grad}\, f = \underset{\sim}{\nabla} f = \underset{\sim}{e}_i \frac{\partial f}{\partial x^i} = \underset{\sim}{e}_i f_{,i},$$

while if $\underset{\sim}{v}$ is a vector, then the *divergence* of $\underset{\sim}{v}$ is given by

$$\text{div}\, \underset{\sim}{v} = \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \frac{\partial v_i}{\partial x^i} = v_{i,i},$$

and the *curl* of $\underset{\sim}{v}$ is given by

$$\text{curl}\, \underset{\sim}{v} = \underset{\sim}{\nabla} \times \underset{\sim}{v} = \varepsilon_{ijk} \frac{\partial v_j}{\partial x^i} \underset{\sim}{e}_k = \varepsilon_{ijk} v_{j,i} \underset{\sim}{e}_k.$$

Finally, the *Laplace* operator, $\nabla^2 = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla}$, can be expressed as

$$\nabla^2 f = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} f = \frac{\partial^2 f}{\partial x^i \partial x^i} = f_{,ii}.$$

Note:

Be careful when writing second order (and higher) derivatives as tensors. Because if you only write

$$\frac{\partial^2 f}{\partial x^{i2}},$$

then it is not clear that you are suppose to sum over the i index (i.e., no obvious repeated index). Whereas from

$$\frac{\partial^2 f}{\partial x^i \partial x^i},$$

it is clear that i is repeated, and hence summed over.

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Exercise 2.8:
Show

$$\underset{\sim}{\nabla} \cdot (\underset{\sim}{a} \times \underset{\sim}{b}) = \underset{\sim}{b} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{a}) - \underset{\sim}{a} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{b}).$$

✠

Answer:

$\square$

Summary

In this lecture, we ...

- introduced the permutation symbol
- expressed some common vector operations in terms of tensors

Coming up

In the next lecture, we ...

- introduce the $\varepsilon - \delta$ identity
- introduce the concepts of metric tensors

Homework Exercise 2.3:

1. Verify the vector identity

$$\underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w}) = \underset{\sim}{v} \cdot (\underset{\sim}{w} \times \underset{\sim}{u}).$$

2. If $f(\underset{\sim}{x})$ is a scalar function and $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are a vectors, then prove the following vector identities
   (a) $\text{curl}(f\underset{\sim}{u}) = \underset{\sim}{\nabla} \times (f\underset{\sim}{u}) = (\underset{\sim}{\nabla} f) \times \underset{\sim}{u} + f(\underset{\sim}{\nabla} \times \underset{\sim}{u})$
   (b) $\underset{\sim}{\nabla} \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = \underset{\sim}{\nabla} \cdot \underset{\sim}{u} + \underset{\sim}{\nabla} \cdot \underset{\sim}{v}$