Review

In the previous lecture, we ...

- introduced the ideas behind continuum mechanics
- introduced the Einstein summation convention

Aims

In this lecture, we will ...

- consider how to classify tensors and basic operations
- introduce the Kronecker delta symbol

Question?

How can a vector $\underset{\sim}{v}$ be expressed using the Einstein summation convention?

Answer:

Question?

For $i, j = 1, \ldots, 3$, are the following two representations of a vector $\underset{\sim}{v}$, namely

$$\underset{\sim}{v} = v_i \underset{\sim}{e}_i \qquad \text{and} \qquad \underset{\sim}{v} = v_j \underset{\sim}{e}_j,$$

different? If so, what is the difference? If not, why are they the same?

Answer:

It is for this reason that the summation index is referred to as a *dummy* index, where the actual "label" of the index is irrelevant.

On the other hand, any other index that is not summed over is called a *free* index, as the index is free to take any valid value over it's range.

Note:
The *range* of an index consists of the set of integers the index is valid for. Usually, the range is a set of n integers ranging from 1 to n.

Note:
If the range of a dummy (or free) index is NOT stated, then the range is assumed to be from 1 to 3.

Example 2.1:

The set of three simultaneous equations

$$y_1 = A_{11}x_1 + A_{12}x_2 + A_{13}x_3,$$
$$y_2 = A_{21}x_1 + A_{22}x_2 + A_{23}x_3,$$
$$y_3 = A_{31}x_1 + A_{32}x_2 + A_{33}x_3,$$

can be written concisely as

$$y_i = A_{ij}x_j.$$

Here, i is a free index, while j is a dummy index. This means we could have used k instead of j, or any other index (except i).

□

Note:
The ranges of i and j are implicitly implied to be $1, \ldots, 3$.

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Question?

What does the tensorial quantity $A_{ii}x_i$ mean?

✠

Answer:

□

Exercise 2.1:

Expand the following tensorial equation

$$A_{ij}A_{ik} = q_{jk},$$

and identify any dummy and free indices, stating their ranges. How many equations are there in total?

✠

Answer:

□

2.2.1 Types of tensors

There are different types of tensors.

To identify these types, we begin by refering to a *tensor system* as any quantity that

1. is represented by a letter with subscripts/superscripts, and
2. obeys certain transformation laws.

For example, the following quantities are tensor systems

$$A^k_{ij}, \quad e^{ijk}, \quad \delta_{ij}, \quad \delta_j{}^i, \quad \delta^i{}_j, \quad A_i,$$

provided they satisfy certain transformation laws (which we will see later).

The total number of subscripts and superscripts determines the *order* of the system.

A system with n indices is called a nth order system. A system with no indices is called a scalar or a zeroth order system.

The *type* of system depends upon the number of subscripts and superscripts occurring in the system.

For example, A^i_{jk} and B^m_{np} are of the same type because they have the same number of subscripts and superscripts (assuming the ranges of the indices match).

In contrast, A^i_{jk} and C^{gh}_f are different types because one has two superscripts while the other only has one.

Note:
In the use of superscripts one must not confuse "powers" of a quantity with the superscripts.

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For example, if we replace the independent variables (x, y, z) with (x^1, x^2, x^3), then we are letting $y = x^2$, where x^2 is a variable and not a power of x. To write a superscript to a power, use parentheses, i.e., $(x^2)^3$.

Note:
The meaning of superscripts and subscripts (and the types) will be considered later.

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Exercise 2.2:

Determine the order and type (i.e., the number of superscripts/subscripts) of the following tensors.

1. A_i.
2. B_i^{jk}.
3. C_{efghi}^{klm}.

✠

Answer:

□

2.2.2 Basic tensor operations

Tensors can be added (or subtracted) together if they are *exactly* the same type.

For example, the sum of A^i_{jk} and B^i_{jk} is again a system of the same type and is denoted, say, by

$$C^i_{jk} = A^i_{jk} + B^i_{jk}.$$

However, the sum of A^i_{jk} and D^{jk}_i is not possible.

The product of two systems is obtained by multiplying each component of the first system with each component of the second system. Such a product is called an *outer product*.

The order of the resulting system is the sum of the orders of the two systems involved in forming the product.

For example, if A^i_j is a second order tensor and B^{lmn} is a third order tensor, with all indices ranging from 1 to 3, then the resulting product system is fifth order and is denoted by, say,

$$C^{ilmn}_j = A^i_j B^{lmn}.$$

The product system represents 3^5 terms constructed from all possible products of the components from A^i_j and B^{lmn}.

The operation of *contraction* occurs when a lower index is set equal to an upper index and the summation index is invoked.

For example, for the fifth order system C^{ilmn}_j, if we set $i = j$ and sum, then we have

$$C^{jlmn}_j = C^{1lmn}_1 + C^{2lmn}_2 + C^{3lmn}_3 = C^{lmn},$$

where C^{lmn} is the resulting third order system.

When contraction occurs, the order of the resulting system is always 2 less than the original system.

Note:
It is usually not possible to perform contraction on two subscript/superscript indices.

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A system is *symmetric* in two of its indices, if the components are unchanged when the indices are interchanged.

For example, the third order tensor T_{ijk} is symmetric in the indices i and k, if

$$T_{ijk} = T_{kji}, \qquad \text{for all values of } i, j \text{ and } k.$$

A system is *skew-symmetric* in two of its indices, if the components change sign when the indices are interchanged.

For example, the fourth order tensor T_{ijkl} is skew-symmetric in the indices j and l, if

$$T_{ijkl} = -T_{ilkj}, \qquad \text{for all values of } i, j, k \text{ and } l.$$

Exercise 2.3:

For the stated tensorial operations, determine if it is valid, and if so, what is the resulting tensor.

Answer:

1. $A_i + B_i$
2. $A_i + B^i$
3. $A_i + B_j$
4. $A_i B^j$
5. $A_i B^i$
6. $A_i B^{jkl} + C_i^j D^{kl}$
7. $A_i B^{jkl} + C_i^j D^{ki}$

□

2.2.3 Special tensors

In writing and manipulating tensors, and equations involving tensors, two special tensors are important to know.

The first is the *Kronecker delta*, δ_{ij}, tensor.

Definition 2.1:

The Kronecker delta, δ_{ij}, is a tensor of order 2 and is defined by

$$\delta_{ij} = \delta_i^j = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

□

Example 2.2:

$$\begin{aligned} \delta_{ij}\delta_{jk} &= \delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k}, \\ &= \begin{cases} 0, & \text{if } i \neq k, \\ 1, & \text{if } i = k, \end{cases} \\ &= \delta_{ik}. \end{aligned}$$

□

Exercise 2.4:

Simplify the following:

1. $\delta_{2m} A_{mj}$.
2. $\delta_{im}\delta_{mn}\delta_{nj}$.

✠

Answer:

□

Note:
The "$\underline{\ }$" under the indices indicates that no summation is applied to the repeated indices.

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Summary

In this lecture, we ...

- considered how to classify tensors and basic operations
- introduced the Kronecker delta symbol

Coming up

In the next lecture, we ...

- introduce the permutation symbol
- express some common vector operations in terms of tensors

Homework Exercise 2.2:

1. Simplify the following tensor systems.
   (a) $a_{ij}\delta_{in}$
   (b) $A_i B_j \delta_{ji} - B_m A_n \delta_{mn}$
   (c) $A_i^j \delta_i^k \delta_{kj}$

2. Assume the third order tensor a_{prs}, for $p, r, s = 1, \ldots, 3$, is completely skew-symmetric in all of its indices. This means that

$$a_{prs} = -a_{psr} = a_{spr} = -a_{srp} = a_{rsp} = -a_{rps}.$$

   Show that 21 of the 27 elements are zero, where the 6 nonzero elements satisfy the above equations when $(p, r, s) = (1, 2, 3)$.