Review

In the previous lecture, we . . .

- introduced Hooke's Law for an elastic solid
- introduced elastic symmetries

Aims

In this lecture, we will . . .

- rewrite Hooke's Law in terms of stress
- consider some examples of Hooke's Law

5.3.1 Elastic moduli for isotropic media

In the previous lecture, we found that the generalized Hooke's Law for isotropic materials is given by

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij},$$

where ϵ_{ij} is the infinitesimal strain tensor, σ_{ij} is the stress tensor and μ and λ are called Lamé's constants.

In the form above, Hooke's Law expresses the stress in terms of the strain. However, the physical meaning of Lamé's constants is not easy to obtain.

Here, we will rewrite Hooke's Law to express the strain in terms of the stress, which results in physically meaningful constants.

To do this, consider the case when $j = i$, namely

$$\sigma_{\underline{i}\,\underline{i}} = \lambda\epsilon_{kk} + 2\mu\epsilon_{\underline{i}\,\underline{i}}, \quad \text{(no summation over } i\text{)}$$

so that summing the values for $i = 1$, $i = 2$ and $i = 3$ together, gives

$$\sigma_{11} + \sigma_{22} + \sigma_{33} = \lambda\epsilon_{kk} + \lambda\epsilon_{kk} + \lambda\epsilon_{kk} + 2\mu\epsilon_{11} + 2\mu\epsilon_{22} + 2\mu\epsilon_{33},$$

which becomes

$$\sigma_{kk} = 3\lambda\epsilon_{kk} + 2\mu\epsilon_{kk},$$

$$\text{i.e.,} \qquad \sigma_{kk} = (3\lambda + 2\mu)\epsilon_{kk}.$$

Hence, solving for ϵ_{kk}, gives

$$\epsilon_{kk} = \frac{\sigma_{kk}}{3\lambda + 2\mu}.$$

Thus, the generalized Hooke's Law for isotropic materials become

$$\sigma_{ij} = \frac{\lambda}{3\lambda + 2\mu}\,\sigma_{kk}\,\delta_{ij} + 2\mu\epsilon_{ij}.$$

Thus, upon solving for ϵ_{ij}, gives

$$\begin{aligned}
\epsilon_{ij} &= \frac{1}{2\mu}\,\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\,\sigma_{kk}\,\delta_{ij}, \\
&= \frac{1}{\mu(3\lambda + 2\mu)}\left[\frac{3\lambda + 2\mu}{2}\,\sigma_{ij} - \frac{\lambda}{2}\,\sigma_{kk}\,\delta_{ij}\right], \\
&= \frac{1}{\mu(3\lambda + 2\mu)}\left[\frac{\lambda + 2(\lambda + \mu)}{2}\,\sigma_{ij} - \frac{\lambda}{2}\,\sigma_{kk}\,\delta_{ij}\right], \\
&= \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\left[\left\{\frac{\lambda}{2(\lambda + \mu)} + 1\right\}\sigma_{ij} - \frac{\lambda}{2(\lambda + \mu)}\,\sigma_{kk}\,\delta_{ij}\right].
\end{aligned}$$

Hence, if we define

$$E = \text{Young's modulus} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu},$$

$$\nu = \text{Poisson's ratio} = \frac{\lambda}{2(\lambda + \mu)},$$

then the generalized Hooke's Law for elastic isotropic materials can be written as

$$\epsilon_{ij} = \frac{1+\nu}{E}\,\sigma_{ij} - \frac{\nu}{E}\,\sigma_{kk}\delta_{ij}, \tag{5.2}$$

or, alternatively

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\epsilon_{ij} + \frac{\nu}{1-2\nu}\,\epsilon_{kk}\,\delta_{ij}\right]. \tag{5.3}$$

Therefore, we have expressed the generalized Hooke's Law in terms of two constants, whose physical meaning can be readily obtained.

For example, for simple tension,

$$\sigma_{11} = T, \quad \text{while all other } \sigma_{ij} = 0,$$

and from (5.2), we find

$$\epsilon_{11} = \frac{T}{E}, \quad \epsilon_{22} = \epsilon_{33} = -\nu \, \frac{T}{E}.$$

From the above equations, the quantity

$$E = \frac{T}{\epsilon_{11}}$$

represents the ratio of the tensile stress T to extension ϵ_{11} produced by the stress T.

Further, we also find

$$\nu = \left|\frac{\epsilon_{22}}{\epsilon_{11}}\right| = \left|\frac{\epsilon_{33}}{\epsilon_{11}}\right|,$$

so that ν denotes the ratio of contraction of the linear elements perpendicular to the axis of extension.

It is easy to verify that Lamé's constants λ and μ can be expressed in terms of Young's modulus E and Poisson's ratio ν as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}.$$

Note:

If $(x^1, x^2, x^3) = (x, y, z)$ then

$$\sigma_{11} = \sigma_{xx}, \quad \sigma_{12} = \sigma_{xy}, \quad \sigma_{13} = \sigma_{xz}, \quad \ldots, \text{ etc.}$$

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Example 5.2:

Consider the stress state:

$$\begin{aligned}\sigma_{xx} &= k[xy - \nu(x^2 - y^2)],\\ \sigma_{yy} &= k[yx - \nu(y^2 - x^2)],\\ \sigma_{zz} &= kxy,\\ \sigma_{xy} &= \frac{k}{2}[4\nu xy - x^2 - y^2],\\ \sigma_{yz} &= \sigma_{xz} = 0.\end{aligned}$$

Determine the strain components according to the generalized Hooke's Law for isotropic materials.

□

Answer:

We know

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}.$$

Thus, we first consider σ_{kk}, i.e.,

$$\begin{aligned}\sigma_{kk} &= \sigma_{11} + \sigma_{22} + \sigma_{33},\\ &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz},\\ &= 3kxy - \nu(x^2 - y^2) - \nu(y^2 - x^2) = 3kxy.\end{aligned}$$

Hence,

$$\begin{aligned}\epsilon_{11} = \epsilon_{xx} &= \frac{1+\nu}{E}\,\sigma_{11} - \frac{\nu}{E}\,\sigma_{kk}\delta_{11},\\ &= \frac{1+\nu}{E}k[xy - \nu(x^2 - y^2)] - \frac{\nu}{E}(3kxy)(1),\\ &= \frac{1-2\nu}{E}kxy - \frac{1+\nu}{E}\nu(x^2 - y^2).\\ \epsilon_{22} = \epsilon_{yy} &= \frac{1+\nu}{E}\,\sigma_{22} - \frac{\nu}{E}\,\sigma_{kk}\delta_{22},\\ &= \frac{1-2\nu}{E}kxy - \frac{1+\nu}{E}\nu(y^2 - x^2).\end{aligned}$$

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\begin{aligned}
\epsilon_{33} = \epsilon_{zz} &= \frac{1+\nu}{E}\,\sigma_{33} - \frac{\nu}{E}\,\sigma_{kk}\delta_{33}, \\
&= \frac{1-2\nu}{E} kxy. \\
\epsilon_{12} = \epsilon_{xy} &= \frac{1+\nu}{E}\,\sigma_{12} - \frac{\nu}{E}\,\sigma_{kk}\delta_{12}, \\
&= \frac{1+\nu}{E}\frac{k}{2}[4\nu xy - x^2 - y^2] - \frac{\nu}{E}.3kxy.0, \\
&= \frac{k(1+\nu)}{2E}(4\nu xy - x^2 - y^2). \\
\epsilon_{13} = \epsilon_{xz} &= 0. \\
\epsilon_{23} = \epsilon_{yz} &= 0.
\end{aligned}
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Therefore, we have found the strain components according to the generalized Hooke's Law for isotropic materials.

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Exercise 5.1:

Determine the strain components according to the generalized Hooke's Law for isotropic materials of the following Cartesian stress distribution.

$$\sigma_{11} = \kappa x^2 + \gamma xy,$$
$$\sigma_{22} = \kappa xy - \gamma y^2,$$
$$\sigma_{12} = (\kappa + \gamma)xy,$$
$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0,$$

where κ and γ are constants.

Answer:

Exercise 5.2:

Given the Cartesian strains in Exercise 5.1, namely

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}x(\kappa x + \gamma y) - \frac{\nu}{E}y(\kappa x - \gamma y),\\
\epsilon_{yy} &= \frac{1}{E}y(\kappa x - \gamma y) - \frac{\nu}{E}x(\kappa x + \gamma y),\\
\epsilon_{zz} &= -\frac{\nu}{E}[\kappa x(x + y) + \gamma y(x - y)],\\
\epsilon_{xy} &= \frac{1+\nu}{E}(\kappa + \gamma)xy,\\
\epsilon_{xz} &= \epsilon_{yz} = 0,
\end{aligned}$$

then determine if the stress distribution that caused these strains is admissible. (Hint: Check the compatibility equations)

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Note:

In terms of (x, y, z), the compatibility equations are:

$$\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \epsilon_{zx}}{\partial z \partial x},$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z},$$

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y},$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} - \frac{\partial \epsilon_{xy}}{\partial z}\right),$$

$$\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} - \frac{\partial \epsilon_{yz}}{\partial x}\right),$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{zx}}{\partial y}\right).$$

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Answer:

□

Note:

If the stress, or strain, distribution is inadmissible, then it means that the stress and strain distributions are not valid for a continuum material.

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Summary

In this lecture, we . . .

- rewrote Hooke's Law in terms of stress
- considered some examples of Hooke's Law

Coming up

In the next lecture, we will . . .

- introduce nanomechanics
- introduce carbon nanotubes

Homework Exercise 5.2:

1. Determine if the following stress distribution is admissible, where the strains are given by the generalized Hooke's Law for isotropic materials.

$$\sigma_{xx} = \kappa x + \gamma y,$$
$$\sigma_{yy} = \gamma x - \kappa y,$$
$$\sigma_{zz} = 0,$$
$$\sigma_{xy} = Az,$$
$$\sigma_{yz} = \sigma_{xz} = 0.$$

2. Determine a quadratic stress distribution that is admissible, where the strains are given by the generalized Hooke's Law for isotropic materials.