

## Review

In the previous lecture, we ...

- expressed the Law's of Motion in a more useful representation
- introduced principal directions and stresses

## Aims

In this lecture, we will ...

- introduce Hooke's Law for an elastic solid
- introduce elastic symmetries

## 5 Constitutive equations

Previously, we have considered the mathematical description of stress, strain and deformation of continuous materials.

We have also developed a number of general theorems applicable to all materials that can be represented with sufficient accuracy as a continuous body.

We now consider equations characterizing an individual material and its reaction to applied loads; such equations are called *constitutive equations*, since they describe the macroscopic behaviour resulting from the internal constitution of the material.

But materials, especially in the solid state, behave in such complex ways when the entire range of possible temperatures and deformations is considered that it is not feasible to write down one equation, or even a set of equations, to accurately describe a real material over its entire range of behaviour.

Instead, separate equations describing various kinds of *ideal responses* are formulated, each of which is a mathematical formulation designed to approximate physical observations of a real material's response over a suitably restricted range.

Here, we concentrate on the characterization of elastic solids.

### 5.1 Hooke's Law

An *ideal elastic solid*, or a Hookean solid, is a body formed of a material that recovers its original form completely upon removal of the forces causing the deformation.

For such solids, there is a one-to-one relationship between the state of stress and the state of strain, for a given temperature.

The ideal material most commonly assumed for stress analysis in structure and machine parts obeys Hooke's Law, which in a uniaxial stress situation takes the form

$$\sigma_{xx} = E\epsilon_{xx},$$

where  $E$  is Young's modulus of elasticity.

This version of Hooke's Law expresses a linear relationship between axial stress and strain.

For pure shear, which closely approximates the twisting of a thin-walled circular tube, where the shear stress is assumed uniformly distributed over the cross-section, the comparable Hooke's Law is

$$\tau = \mu\gamma,$$

where  $\mu$  is the shear modulus,  $\tau$  is the shear stress in the circumferential direction, and  $\gamma$  is the angle of change for elements initially in the axial and circumferential directions.

## 5.2 Generalized Hooke's Law

In a more general idealized elastic body we merely say that the stress depends on the strain, or deformation, from a certain natural (unstressed) state. Namely,

$$\text{stress tensor} = f(\text{deformation tensor}).$$

The deformation gradient used in the argument above for the function may be one of the finite-strain tensors or the deformation gradient  $\tilde{F}$ .

The functional relationship between the two tensors may be non-linear, but it must express a one-to-one relationship between the stress and the strain.

Now, if the function  $f$  is expanded in power series in powers of  $\epsilon_{ij}$ , where non-linear terms are neglected, then we get

$$\sigma_{ij} = c_{ijkl} \epsilon_{kl},$$

where  $c_{ijkl}$  is the *Cauchy stress tensor* and  $\epsilon_{kl}$  is the small-strain tensor. This equation is valid provided the displacement gradients are small when compared with unity.

The above (nine) equations are called the generalized Hooke's Law, and they are used in all developments of the theory of linear elasticity.

While there are 81 constants contained in the above nine equations, the  $c_{ijkl}$ 's (referred to as the elastic moduli) are not all independent.

Since  $\sigma_{ij}$  is symmetric, then from the generalized Hooke's Law we must have

$$c_{ijkl} = c_{jikl}.$$

Further, since  $\epsilon_{kl}$  is symmetric in respect to the dummy indices  $kl$ , we must have

$$c_{ijkl} = c_{jilk}.$$

Hence, the set of 81 constants have been reduced to 36.

### 5.2.1 Matrix form of the generalized Hooke's Law

Because of the symmetry of both the stress tensor and the strain tensor, the nine equations making up the generalized Hooke's Law really represents only six independent equations, each containing six terms, for a total of only 36 elastic constants (moduli) instead of 81.

These equations are conveniently represented as a matrix equation expressing a six-element column matrix of stresses in terms of a six-element column matrix of strains, as follows:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{pmatrix}.$$

In order to avoid dealing with double sums, we introduce the following notation for the elements of the two column matrices  $\boldsymbol{\sigma}$  and  $\boldsymbol{\epsilon}$ , namely

$$\begin{aligned} \sigma_1 &= \sigma_{11}, & \sigma_2 &= \sigma_{22}, & \sigma_3 &= \sigma_{33}, & \sigma_4 &= \sigma_{23}, & \sigma_5 &= \sigma_{31}, & \sigma_6 &= \sigma_{12}, \\ \epsilon_1 &= \epsilon_{11}, & \epsilon_2 &= \epsilon_{22}, & \epsilon_3 &= \epsilon_{33}, & \epsilon_4 &= 2\epsilon_{23}, & \epsilon_5 &= 2\epsilon_{31}, & \epsilon_6 &= 2\epsilon_{12}. \end{aligned}$$

Thus, the matrix equation becomes

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{C}} \boldsymbol{\epsilon},$$

which in tensor notation gives

$$\sigma_i = c_{ij} \epsilon_j,$$

where  $\boldsymbol{\mathcal{C}}$  is the  $6 \times 6$  square matrix whose components are the elastic moduli  $c_{ij}$ . The above equation is the generalized Hooke's Law for an anisotropic elastic body.

It is *usual* to assume  $c_{ij} = c_{ji}$ , in which case the number of distinct elastic constants in the generalized Hooke's Law is reduced to 21.

### 5.2.2 Elastic symmetries

If an elastic medium exhibits a geometrical symmetry of internal structure (crystallographic form, regular arrangement of fibres or molecules, etc.), then its elastic properties become identical in certain directions.

However, geometric symmetries are not equivalent to elastic symmetries because there may be certain other directions for which the elastic properties are the same but the geometric ones are not.

If the medium is elastically symmetric in certain directions, then the number of independent constants  $c_{ij}$  is further reduced.

If the medium has an elastic symmetry, then there must exist a transformation of coordinates  $\bar{x}^i = \bar{x}^i(x^1, x^2, x^3)$  such that

$$\bar{c}_{ij} = c_{ij},$$

where  $\sigma_i = c_{ij}\epsilon_j$ , and  $\bar{\sigma}_i = \bar{c}_{ij}\bar{\epsilon}_j$ .

It is obvious that  $c_{ij}$ , in general, depends on the chosen reference frame inasmuch as the stress components  $\sigma_i$  and the strain components  $\epsilon_i$  vary with the choice of coordinate system.

For certain elastic media,  $c_{ij}$  is invariant under a given transformation of coordinates, and it is this invariance that describes the elastic symmetry of the medium under consideration.

### Example 5.1:

Consider a substance elastically symmetric with respect to the  $x^1x^2$ -plane. This symmetry is expressed by the statement that  $c_{ij}$  is invariant under the transformation

$$x^1 = \bar{x}^1, \quad x^2 = \bar{x}^2, \quad x^3 = -\bar{x}^3.$$

The table of direction cosines of this transformation is

	$\bar{x}^1$	$\bar{x}^2$	$\bar{x}^3$
$x^1$	1	0	0
$x^2$	0	1	0
$x^3$	0	0	-1

so that using the transformation law for second order tensors (recalling that  $\sigma_i$  and  $\epsilon_i$  are actually second order tensors), ...

..., we find

$$\begin{aligned} \bar{\sigma}_i &= \sigma_i & \text{and} & & \bar{\epsilon}_i &= \epsilon_i, & (i = 1, 2, 3, 6) \\ \bar{\sigma}_i &= -\sigma_i & \text{and} & & \bar{\epsilon}_i &= -\epsilon_i. & (i = 4, 5) \end{aligned}$$

Thus, due to the symmetry (i.e.,  $\bar{c}_{ij} = c_{ij}$ ), we have

$$\bar{\sigma}_i = \sum_{j=1}^6 \bar{c}_{ij} \bar{\epsilon}_j = \sum_{j=1}^6 c_{ij} \bar{\epsilon}_j,$$

which upon comparing with the expression given by

$$\sigma_i = \sum_{j=1}^6 c_{ij} \epsilon_j,$$

for  $i = 1$ , shows that  $c_{14} = c_{15} = 0$ .

Similarly, by comparing  $\bar{\sigma}_i$  with  $\sigma_i$ , for  $i = 2, 3, 4, 5, 6$ , we find

$$\begin{aligned} c_{24} &= c_{25} = c_{34} = c_{35} = c_{64} = c_{65} = 0, \\ c_{41} &= c_{42} = c_{43} = c_{46} = c_{51} = c_{52} = c_{53} = c_{56} = 0. \end{aligned}$$

Thus, for an elastic material with an elastic symmetry around the  $x^1x^2$ -plane, the matrix of elastic coefficients is given by

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{21} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{31} & c_{32} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{54} & c_{55} & 0 \\ c_{61} & c_{62} & c_{63} & 0 & 0 & c_{66} \end{pmatrix}.$$

Therefore, there are now 13 independent elastic constants ( $c_{ij} = c_{ji}$ ).

□

### 5.2.3 Orthotropic symmetry

Materials such as wood have three mutually orthogonal planes of elastic symmetry and are said to be *orthotropic*.

When considering such materials, we choose the axes of the coordinates so that the coordinate planes coincide with the planes of elastic symmetry.

Besides the symmetry with respect to the  $x^1x^2$ -plane, the elastic constants  $c_{ij}$  must also be invariant under the transformation of coordinates defined by the following table of direction cosines:

	$\bar{x}^1$	$\bar{x}^2$	$\bar{x}^3$
$x^1$	-1	0	0
$x^2$	0	1	0
$x^3$	0	0	1

This change of coordinates is a reflection in the  $x^2x^3$ -plane, and leaves  $\sigma_i$  and  $\epsilon_i$  unchanged with the following exceptions:

$$\bar{\sigma}_i = -\sigma_i \quad \text{and} \quad \bar{\epsilon}_i = -\epsilon_i, \quad i = 5, 6.$$

Thus, from

$$\sigma_1 = c_{11} \epsilon_1 + c_{12} \epsilon_2 + c_{13} \epsilon_3 + c_{16} \epsilon_6, \quad (c_{14} = c_{15} = 0)$$

$$\bar{\sigma}_1 = c_{11} \bar{\epsilon}_1 + c_{12} \bar{\epsilon}_2 + c_{13} \bar{\epsilon}_3 + c_{16} \bar{\epsilon}_6,$$

we obtain  $c_{16} = 0$ .

Similarly, by considering the transformed expressions for other components of the stress tensor, we find that

$$c_{26} = c_{36} = c_{45} = c_{54} = c_{61} = c_{62} = c_{63} = 0.$$

Thus, for orthotropic media the matrix of  $c_{ij}$  becomes

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}.$$

**Note:**

Elastic symmetry in the  $x^1x^2$ -plane and in the  $x^2x^3$ -plane implies elastic symmetry in the  $x^1x^3$ -plane. ♪

which only involves 9 independent constants ( $c_{ij} = c_{ji}$ ).

### 5.3 Homogeneous isotropic media

In an isotropic material, the elastic properties are independent of the orientation of the coordinate axis. In particular, the elastic constants  $c_{ij}$  must remain invariant for any choice of coordinate system.

This implies that there are only two independent elastic constants, which can be proved by the following steps:

First, we rotate the  $(x^1, x^2, x^3)$  coordinate system through a right angle about the  $x^1$ -axis. Upon considering the transformed coordinates in exactly the same way as in the previous section (i.e., as a symmetry), we find ...

... we find

$$c_{12} = c_{13}, \quad c_{31} = c_{21}, \quad c_{32} = c_{23}, \quad c_{33} = c_{22}, \quad c_{66} = c_{55}.$$

Secondly, a rotation of axes through a right angle about the  $x^3$ -axis leads to the relations

$$c_{12} = c_{21}, \quad c_{22} = c_{11}, \quad c_{32} = c_{13}, \quad c_{31} = c_{23}, \quad c_{44} = c_{55}.$$

Finally, we rotate the axes through an angle of 45 degrees about the  $x^3$ -axis, from which we obtain

$$c_{22} = c_{11}, \quad c_{13} = c_{23} = c_{21} = c_{12}, \quad c_{44} = \frac{1}{2}(c_{11} - c_{12}).$$

Thus, if we let  $c_{44} = \mu$  and  $c_{12} = \lambda$ , then the matrix of elastic constants become ...

$$\begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}.$$

Thus, for isotropic media, the fourth order tensor  $\underline{c}$  can be expressed in indicial notation as

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

where the constants  $\lambda$  and  $\mu$  were introduced by G. Lamé, and as such are called Lamé constants.

Hence, the generalized Hooke's Law for *isotropic materials* can be expressed as

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}, \quad (5.1)$$

where  $\epsilon_{kk} = \nabla \cdot \underline{u}$  and  $\delta_{ij}$  is the Kronecker delta.

We have shown that the stress-strain law for isotropic materials involve no more than two elastic constants. The fact that there is no further possible reduction is physically obvious, but an analytical proof of this can be constructed by using the properties of the isotropic tensors.

**Note:**

From the stress-strain relation (5.1), we have the result that the principal axes of stress are coincident with the principal axes of strain (for a isotropic medium).

Henceforth, no distinction will be made between the principal axes of strain and the principal axes of stress, where such axes will be simply referred to as the principal axes.



## Summary

In this lecture, we ...

- introduced Hooke's Law for an elastic solid
- introduced elastic symmetries

## Coming up

In the next lecture, we will ...

- rewrite Hooke's Law in terms of stress
- consider some examples of Hooke's Law

## Homework Exercise 5.1:

1. Using (5.1), consider the case when  $i = j$ . Show that summing over  $i$  gives

$$\epsilon_{kk} = \frac{\sigma_{kk}}{3\lambda + 2\mu}.$$

2. Using 1. above, show that (5.1) can be expressed in the form

$$\epsilon_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij},$$

where

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}.$$