Review

In the previous lecture, we . . .

- examined the idea of stress
- introduced the Law's of Motion

Aims

In this lecture, we will . . .

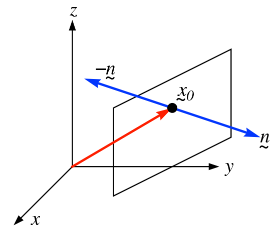
- express the Law's of Motion in a more useful representation
- introduce principal directions and stresses

4.4 Some important results

Before we express the Law's of Motion in a more useful manner, we first need to introduce some important results.

Result 1: Newton's third law of action and reaction

Let $\underset{\sim}{x}_0$ be a fixed point, $\underset{\sim}{x}$ be a Cartesian coordinate frame. Then the stress vector $\underset{\sim}{T}(\underset{\sim}{x}_0, \underset{\sim}{n})$ at $\underset{\sim}{x}_0$ on the surface element whose normal is $\underset{\sim}{n}$ is related to the stress vector $\underset{\sim}{T}(\underset{\sim}{x}_0, -\underset{\sim}{n})$ at $\underset{\sim}{x}_0$ on the surface element whose normal is $-\underset{\sim}{n}$ by the relation



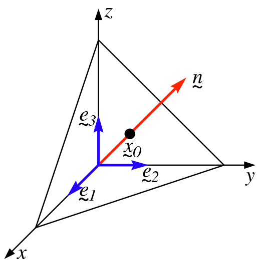
$$\underset{\sim}{T}(\underset{\sim}{x}_0, \underset{\sim}{n}) = -\underset{\sim}{T}(\underset{\sim}{x}_0, -\underset{\sim}{n}).$$

□

Result 2:

Let $\underset{\sim}{x}_0$ be a fixed point, $\underset{\sim}{x}$ be a Cartesian coordinate system with axes x_i and unit normal base vectors $\underset{\sim}{e}_i$.

Then the stress vector on the surface element through $\underset{\sim}{x}_0$ with normal $\underset{\sim}{n}$ is related to the stress vectors on the surfaces through $\underset{\sim}{x}_0$ with normals $\underset{\sim}{e}_i$ according to



$$\underset{\sim}{T}(\underset{\sim}{x}_0, \underset{\sim}{n}) = \underset{\sim}{T}(\underset{\sim}{x}_0, \underset{\sim}{e}_i)\, n_i,$$

where $\underset{\sim}{n} = n_i\, \underset{\sim}{e}_i$.

□

Result 3:

Now, if we consider the decomposition of $\underset{\sim}{T}(\underset{\sim}{x}_0, \underset{\sim}{n})$ along $\underset{\sim}{e}_i$, then we can write

$$\underset{\sim}{T}(\underset{\sim}{x}_0, \underset{\sim}{n}) = T_i(\underset{\sim}{x}_0, \underset{\sim}{n})\, \underset{\sim}{e}_i.$$

Previously, we defined

$$\underset{\sim}{T}(\underset{\sim}{x}_0, \underset{\sim}{e}_i) = \sigma_{ij}\, \underset{\sim}{e}_j,$$

so that

$$\underset{\sim}{T}(\underset{\sim}{x}_0, \underset{\sim}{n}) = \underset{\sim}{T}(\underset{\sim}{x}_0, \underset{\sim}{e}_i)\, n_i = \sigma_{ij}\, \underset{\sim}{e}_j\, n_i = \sigma_{ij}\, n_i\, \underset{\sim}{e}_j = T_j(\underset{\sim}{x}_0, \underset{\sim}{n})\, \underset{\sim}{e}_j.$$

Thus,

$$T_j(\underset{\sim}{x}_0, \underset{\sim}{n}) = \sigma_{ij}\, n_i.$$

□

The above result has 2 important interpretations and uses.

1. The problem of finding an *infinite number* of stress vectors on the infinite number of surfaces through $\underset{\sim}{x}$, defined by the infinite number of choices for normal $\underset{\sim}{n}$, is now reduced to finding the stress vectors on *three* mutually perpendicular surfaces through $\underset{\sim}{x}$.

2. The stress vector on a surface is part of the *prescribed* stress distribution acting on the exterior surface, i.e., $\underset{\sim}{T}(\underset{\sim}{x}, \underset{\sim}{n})$ is a given quantity, where $\underset{\sim}{n}$ is on the exterior. Then the results relates the prescribed information on the *exterior* to the stress vectors acting on the *interior* surface with normals $\underset{\sim}{e}_i$. In these circumstances, the result is used to express a *boundary condition*.

Result 4: Newton's second law of motion

For a body in motion, the stress distribution and motion are related by

$$\frac{\partial \sigma_{ji}}{\partial x^j} + f_i = \rho \ddot{x}^i,$$

where $\sigma_{ij} = \sigma_{ji}$. Further, the body is in equilibrium if and only if

$$\frac{\partial \sigma_{ji}}{\partial x^j} + f_i = 0.$$

□

Proof:

Consider the law of conservation of linear momentum

$$\iiint_B \underset{\sim}{f}\,\mathrm{d}V + \iint_A \underset{\sim}{T}(\underset{\sim}{x}, \underset{\sim}{n})\,\mathrm{d}S = \iiint_B \rho \ddot{\underset{\sim}{x}}\,\mathrm{d}V,$$

using the component form.

Namely,

$$\iiint_B f_i\,\mathrm{d}V + \iint_A T_i(\underset{\sim}{x}, \underset{\sim}{n})\,\mathrm{d}S = \iiint_B \rho \ddot{x}^i\,\mathrm{d}V,$$

where we know

$$T_i(\underset{\sim}{x}, \underset{\sim}{n}) = \sigma_{ji} n_j \quad \Rightarrow \quad \iint_A T_i(\underset{\sim}{x}, \underset{\sim}{n})\,\mathrm{d}S = \iint_A \sigma_{ji} n_j\,\mathrm{d}S.$$

Then, from Green's theorem, i.e.,

$$\iint_A \phi\, n_i\,\mathrm{d}S = \iiint_B \frac{\partial \phi}{\partial x^i}\,\mathrm{d}V,$$

we find

$$\iint_A T_i(\underset{\sim}{x}, \underset{\sim}{n})\,\mathrm{d}S = \iiint_B \frac{\partial \sigma_{ji}}{\partial x^j}\,\mathrm{d}V.$$

Therefore, the conservation law of linear momentum becomes

$$\iiint_B \left(\frac{\partial \sigma_{ji}}{\partial x^j} + f_i - \rho \ddot{x}^i \right)\,\mathrm{d}V \equiv 0, \tag{4.5}$$

which has to be true for every part of the body, so that

$$\frac{\partial \sigma_{ji}}{\partial x^j} + f_i = \rho \ddot{x}^i.$$

Further, at equilibrium $\ddot{x} = 0$, so that the equation becomes

$$\frac{\partial \sigma_{ji}}{\partial x^j} + f_i = 0.$$

Now, apply the law of conservation of moment of momentum to the same body, namely

$$\iiint_B \underset{\sim}{x} \times \underset{\sim}{f}\,\mathrm{d}V + \iint_A \underset{\sim}{x} \times \underset{\sim}{T}(\underset{\sim}{x}, \underset{\sim}{n})\,\mathrm{d}S = \iiint_B \underset{\sim}{x} \times \rho \ddot{\underset{\sim}{x}}\,\mathrm{d}V,$$

which in component form becomes

$$\iiint_B \varepsilon_{ijk}\, x^j f_k \,\mathrm{d}V + \iint_A \varepsilon_{ijk}\, x^j T_k(\underset{\sim}{x}, \underset{\sim}{n})\,\mathrm{d}S = \iiint_B \varepsilon_{ijk}\, x^j \rho \ddot{x}^k \,\mathrm{d}V.$$

Therefore, as $T_k = \sigma_{mk}\, n_m$, then

$$\begin{aligned}\iint_A \varepsilon_{ijk}\, x^j T_k(\underset{\sim}{x}, \underset{\sim}{n})\,\mathrm{d}S &= \iint_A \varepsilon_{ijk}\, x^j\, \sigma_{mk}\, n_m \,\mathrm{d}S,\\ &= \iiint_B \varepsilon_{ijk}\, \frac{\partial}{\partial x^m}\left\{x^j\, \sigma_{mk}\right\}\,\mathrm{d}V,\\ &= \iiint_B \varepsilon_{ijk}\left(\delta_{jm}\sigma_{mk} + x^j \frac{\partial \sigma_{mk}}{\partial x^m}\right)\mathrm{d}V,\\ &= \iiint_B \varepsilon_{ijk}\left(\sigma_{jk} + x^j \frac{\partial \sigma_{mk}}{\partial x^m}\right)\mathrm{d}V,\end{aligned}$$

so that the conservation of moment of momentum becomes

$$\iiint_B \varepsilon_{ijk}\left(x^j f_k + \sigma_{jk} + x^j \frac{\partial \sigma_{mk}}{\partial x^m} - x^j \rho \ddot{x}^k\right)\mathrm{d}V \equiv 0,$$

$$\Rightarrow \quad \iiint_B \varepsilon_{ijk}\, x^j \left(\frac{\partial \sigma_{mk}}{\partial x^m} + f_k - \rho \ddot{x}^k\right)\mathrm{d}V + \iiint_B \varepsilon_{ijk}\, \sigma_{jk}\,\mathrm{d}V \equiv 0,$$

i.e.,

$$\iiint_B \varepsilon_{ijk}\, \sigma_{jk}\,\mathrm{d}V \equiv 0,$$

due to conservation of linear momentum. Now, as B is arbitrary

$$\varepsilon_{ijk}\, \sigma_{jk} = 0,$$

which leads to

$$\sigma_{jk} = \sigma_{kj},$$

or in other words, the stress tensor is symmetric.

□

Result 5:

Recall, for the conservation of mass, we assumed

$$\frac{\mathrm{D}}{\mathrm{D}t}\left\{\iiint_{V_t} \rho(\underset{\sim}{x}, t)\,\mathrm{d}V\right\} = 0,$$

where V_t is the volume of the material at time t, and $\mathrm{D}/\mathrm{D}t$ is called the *material time derivative*, given by

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \frac{\partial x^i}{\partial t}\frac{\partial}{\partial x^i}.$$

Another way to consider the conservation of mass, is to let the mass of a volume element V_t be given by

$$m = \iiint_{V_t} \rho(\underset{\sim}{x}, t)\,\mathrm{d}V,$$

so the rate of change of mass of that volume element becomes

$$\frac{\partial m}{\partial t} = \iiint_{V_t} \frac{\partial \rho}{\partial t}\,\mathrm{d}V.$$

Now, this rate of change must be equivalent to the flux rate of mass crossing the surface S of the volume element V, which is given by

$$I = \iint_S \rho \underset{\sim}{v} \cdot \underset{\sim}{n}\,\mathrm{d}S,$$

where $\underset{\sim}{v}$ is the velocity field and $\underset{\sim}{n}$ is the exterior unit normal to the surface S.

I represents the change of mass leaving or entering the region if $\rho \underset{\sim}{v} \cdot \underset{\sim}{n}$ is positive or negative, respectively.

Thus,

$$\frac{\partial m}{\partial t} = -\iint_S \rho \underset{\sim}{v} \cdot \underset{\sim}{n} \, \mathrm{d}S,$$

so that we obtain the equation

$$\iiint_{V_t} \frac{\partial \rho}{\partial t} \, \mathrm{d}V = -\iint_S \rho \underset{\sim}{v} \cdot \underset{\sim}{n} \, \mathrm{d}S.$$

Now, using Green' theorem, namely

$$\iint_S \phi n_i \mathrm{d}S = \iiint_B \frac{\partial \phi}{\partial x^i} \mathrm{d}V,$$

then the conservation of mass equation becomes

$$\iiint_{V_t} \frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{v}) \, \mathrm{d}V = 0.$$

However, as this integral must hold in any arbitrary system, then we obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{v}) = 0.$$

□

Exercise 4.4:

If

$$\underset{\sim}{\nabla} \cdot (f \underset{\sim}{a}) = \underset{\sim}{a} \cdot \underset{\sim}{\nabla} f + f \underset{\sim}{\nabla} \cdot \underset{\sim}{a},$$

then expand the continuity equation above into indicial notation. Further, express the equation in terms of the material derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i}.$$

What happens when ρ is a constant?

✠

Answer:

□

4.5 Principal directions and stresses

Let $\underset{\sim}{n}$ be an arbitrary unit vector. Then the stress vector on the surface element with normal $\underset{\sim}{n}$ can be written as

$$\underset{\sim}{T}(\underset{\sim}{x}, \underset{\sim}{n}) = \underset{\sim}{N} + \underset{\sim}{S},$$

where $\underset{\sim}{N}$ is the normal stress (tensile stress) and $\underset{\sim}{S}$ is the shear stress (tangential to the surface).

Now, considering the projection, we can write

$$\begin{aligned}
& \underset{\sim}{N} = (\underset{\sim}{T}(\underset{\sim}{x}, \underset{\sim}{n}) \cdot \underset{\sim}{n}) \underset{\sim}{n} = N \underset{\sim}{n}, && \underset{\sim}{N} \parallel \underset{\sim}{n} \\
\Rightarrow \quad & N = T_i(\underset{\sim}{x}, \underset{\sim}{n}) \, n_i, && \\
& \quad = \sigma_{ji} \, n_j \, n_i = \sigma_{ij} \, n_j \, n_i, && (\sigma_{ij} = \sigma_{ji})
\end{aligned}$$

and $\underset{\sim}{S} = \underset{\sim}{T}(\underset{\sim}{x}, \underset{\sim}{n}) - \underset{\sim}{N}$.

If $\underset{\sim}{S} = \underset{\sim}{0}$, i.e., there is no shear stress, on a surface element with normal $\underset{\sim}{n}$, then

$$\underset{\sim}{T}(\underset{\sim}{x}, \underset{\sim}{n}) = \underset{\sim}{N} = \sigma \underset{\sim}{n},$$

where $\underset{\sim}{n}$ is called the *principal direction* (or principal axes), and σ is the corresponding *principal stress*. In component form, we get

$$T_i(\underset{\sim}{x}, \underset{\sim}{n}) = \sigma n_i.$$

Also, generally, we have

$$T_i(\underset{\sim}{x}, \underset{\sim}{n}) = \sigma_{ji}\, n_j = \sigma_{ij}\, n_j,$$

so that we find

$$\sigma_{ij}\, n_j = \sigma n_i \quad \Rightarrow \quad (\sigma_{ij} - \sigma\delta_{ij})n_j = 0.$$

This leads to solving the eigenvalue problem

$$|\sigma_{ij} - \sigma\delta_{ij}| = 0,$$

i.e.,

$$\begin{vmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{vmatrix} = 0.$$

When this determinant is expanded, it gives a cubic equation for σ, namely

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0, \tag{4.6}$$

where

$$I_1 = \mathrm{tr}(\sigma_{ij}) = \sigma_{kk}, \qquad I_2 = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}), \qquad I_3 = |\sigma_{ij}|.$$

Alternatively, we can write

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33},$$

$$I_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix},$$

$$I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix},$$

recalling that $\sigma_{ij} = \sigma_{ji}$.

I_1, I_2 and I_3 are called the *fundamental scalar invariants* of the stress tensor, and they do *not* depend on the choice of the coordinate system.

For a real symmetric stress tensor σ_{ij}, three real eigenvalues exist, namely σ_1, σ_2 and σ_3 (which are the solutions of (4.6)), and are called the principal stresses.

The corresponding eigenvectors define the principal directions (principal axes), and if the three principal stresses are distinct, then the principal axes are mutually perpendicular.

Note:

1. If the eigenvalues are all distinct, then the corresponding eigenvectors are mutually orthogonal.
2. For every real symmetric tensor, there exists at least one triad of mutually perpendicular eigenvectors. These vectors define the principal directions of the tensor.

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In terms of the principal stresses, the stress invariants become

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3,$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1,$$
$$I_3 = \sigma_1\sigma_2\sigma_3.$$

Note:

1. The principal stresses are usually denoted such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$.
2. If $\sigma_1 = \sigma_2 = \sigma_3 = \sigma = -p$, where p represents the hydrostatic pressure and σ is the hydrostatic stress, then as $I_1 = \sigma_{kk}$ we have the so-called hydrostatic stress state

$$\sigma = -p = \frac{1}{3}\sigma_{kk}.$$

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Exercise 4.5:

Let the stress of state at a point in a continuum be given by

$$\sigma_{ij} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the principal stresses and the stress invariants. ✠

Answer:

□

Summary

In this lecture, we ...

- expressed the Law's of Motion in a more useful representation
- introduced principal directions and stresses

Coming up

In the next lecture, we will ...

- introduce Hooke's Law for an elastic solid
- introduce elastic symmetries

Homework Exercise 4.3:

1. Consider the conservation law of angular momentum, i.e.,

$$\iiint_B \underset{\sim}{x} \times \underset{\sim}{f} \, \mathrm{d}V + \iint_A \underset{\sim}{x} \times \underset{\sim}{T} (\underset{\sim}{x}, \underset{\sim}{n}) \, \mathrm{d}S = \iiint_B \underset{\sim}{x} \times \rho \underset{\sim}{\ddot{x}} \, \mathrm{d}V.$$

   Upon assuming the conservation of linear momentum, show that $\sigma_{ij} = \sigma_{ji}$. What does this mean?

2. Let the stress of state at a point in a continuum be given by

$$\sigma_{ij} = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Find the principal stresses and the stress invariants.