

## Review

In the previous lecture, we ...

- introduced external and internal forces
- introduced traction

## Aims

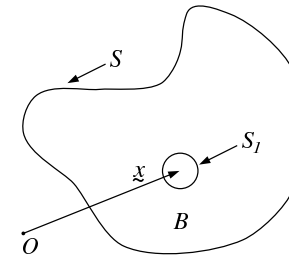
In this lecture, we will ...

- examine the idea of stress
- introduce the Law's of Motion

## 4.2 Stress

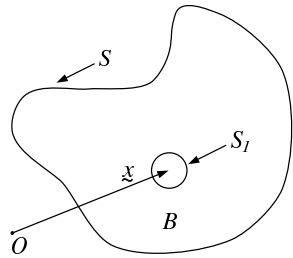
### 4.2.1 Ideas of stress

When we consider a particular deformation of a particular material, we usually know the force per unit area, i.e., the stress vector, on the exterior surface  $S$  of the body  $B$ .



In this situation, we want to find the distribution of the force per unit area (i.e., the stress vector) on the interior surface  $S_1$ .

But the question is, on which interior surfaces are the internal force (stress) distribution of interest? Through any point  $\tilde{x}$  in  $B$ , there exists an *infinite* number of surfaces, each having a different orientation with a different stress distribution.

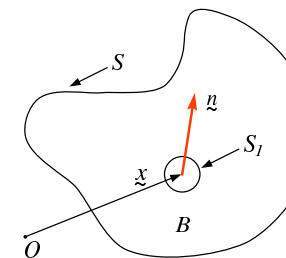


A vital point to understand is that there can be a *different* stress vector on each area element  $dS_1$  through  $\tilde{x}$ .

On some area elements the force intensity (stress) is large, while on others it is small.

Thus, the stress vector  $\tilde{T}$  depends *not only* on the point  $\tilde{x}$ , but *also* on the direction of the surface through  $\tilde{x}$  on which it acts.

The direction of an element of the surface area  $S_1$  at  $\tilde{x}$  is uniquely specified by giving the outer unit normal  $\tilde{n}$  at  $\tilde{x}$ .



Hence,

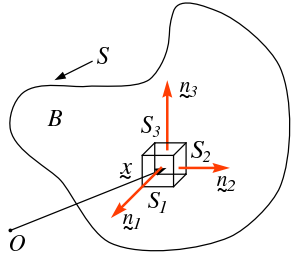
$$\tilde{T} = \tilde{T}(\tilde{x}, \tilde{n}).$$

Our aim is to find the sum of all  $\tilde{T}(\tilde{x}, \tilde{n})$  for each fixed  $\tilde{x}$  as  $\tilde{n}$  takes on all directions. This is called the *local state of stress* at  $\tilde{x}$ .

In other words, for a fixed  $\tilde{x}$ , we want to find  $\tilde{T}(\tilde{x}, \tilde{n})$  for all possible choices of  $\tilde{n}$ .

### 4.2.2 Cartesian components of stress

The task of finding the local state of stress at  $\underline{x}$  can be simplified if we first introduce the stress vectors acting on three mutually perpendicular surface elements through  $\underline{x}$ .



Let  $\underline{x}$  be a fixed point in  $B$ . For simplicity, we write  $\underline{T}$  at  $\underline{x}$  as

$$\underline{T}(\underline{x}, \underline{n}) = \underline{T}(\underline{n}).$$

Let  $\underline{T}(\underline{e}_i)$  denote the stress vectors on three mutually perpendicular surfaces having outer unit normal  $\underline{n}_i = \underline{e}_i$  through  $\underline{x}$ .

We introduce the following special notation for writing the stress vector in components form:

$\underline{\sigma}$  – is the stress tensor with components  $\sigma_{ij}$ .

$\sigma_{ij}$  – are the set of nine scalars giving the components of the three stress vectors  $\underline{T}(\underline{e}_i)$  with respect to the coordinate system  $\underline{x}$ .

$i$  – is the first subscript, which indicates the direction of the normal to the surface.

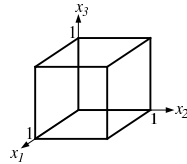
$j$  – is the second subscript, which indicates the direction of the component  $\underline{T}(\underline{e}_i)$ .

Thus,

$$\underline{T}(\underline{e}_i) = \sigma_{ij} \underline{e}_j = \sigma_{i1} \underline{e}_1 + \sigma_{i2} \underline{e}_2 + \sigma_{i3} \underline{e}_3.$$

#### Exercise 4.3:

Given an infinitesimal unit cube of material,



sketch the location and direction of all nine stress components  $\sigma_{ij}$ , and note the direction of the normals  $\underline{e}_i$ .



**Answer:**



In summary, when  $\underline{n} = \underline{e}_i$ ,

$$\underline{T}(\underline{e}_i) = \sigma_{ij} \underline{e}_j, \quad \text{where } \sigma_{ij} = \sigma_{ji}.$$

Physically, we see that:

$\sigma_{ii}$  – is a normal stress, where

$$\sigma_{ii} \begin{cases} > 0 & \text{tension} \\ < 0 & \text{compression} \end{cases}$$

$\sigma_{ij}, i \neq j$  – is a shear stress

**Note:**

In some engineering applications, the stress tensor is assumed to be positive in compression.



The stress tensor  $\underline{\sigma}$  is a second order tensor, which transforms according to the transformation laws of tensors.

Thus, for a new coordinate system  $\bar{x}$  with  $\bar{e}_i$ , the tensor  $\underline{T}(\bar{e}_i)$  has nine new components  $\bar{\sigma}_{ij}$  such that

$$\underline{T}(\bar{e}_i) = \bar{\sigma}_{ij} \bar{e}_j,$$

where

$$\bar{\sigma}_{ij} = \sigma_{mn} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j}.$$

## 4.3 Laws of motion

In continuum mechanics, there are two types of equations.

- (i) Conservation laws of motion that *all* materials must satisfy.
- (ii) Constitutive equations that *changes for each* individual material.

All continuum materials satisfy the conservation laws, irrespective of material, deformation or anything else.

We are going to derive three conservation laws of motion, namely

- (i) conservation of mass,
- (ii) conservation of linear momentum, and
- (iii) conservation of angular (moment of) momentum.

### 4.3.1 Conservation of mass

This law is based on the assumption that the total mass of a fixed part of material (i.e., the material volume) should remain constant at all times. Thus,

$$\frac{D}{Dt} \left\{ \iiint_{V_t} \rho(\underline{x}, t) dV \right\} = 0, \quad (4.1)$$

where  $V_t$  is the volume of the material at time  $t$ , and  $D/Dt$  is called the *material time derivative*, and is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i}.$$

Since  $\underline{x} = \underline{x}(\underline{X}, t)$  and  $\underline{x} = \underline{X}$  at  $t = 0$ , then we define

$$\rho_0(\underline{X}) = \rho(\underline{x}, 0) = \rho(\underline{X}, 0),$$

so that from (4.1) we can say

$$\iiint_{V_0} \rho_0(\underline{X}) dV_0 \equiv \iiint_{V_t} \rho(\underline{x}, t) dV,$$

where  $V_0$  is the volume of the material at time  $t = 0$ .

Using  $dV = JdV_0$ , we can rewrite the above equation to give

$$\iiint_{V_0} (\rho_0(\underline{X}) - \rho(\underline{x}, t) J) dV_0 \equiv 0$$

for an arbitrary  $V_0$ , where  $J$  is the Jacobian determinant.

But as this equation must hold for any arbitrary volume  $V_0$ , then we must set the integrand equal to zero in general, i.e.,

$$\rho_0(\underline{X}) = \rho(\underline{x}, t) J. \quad (4.2)$$

We will come back later to express the conservation of mass equation in a “nicer” form, but at the moment we will just utilize the result (4.2), i.e.,  $\rho_0 = \rho J$ .

### 4.3.2 Conservation of linear momentum

Based upon Newton’s second law, i.e.,

$$\text{force} = \text{mass} \times \text{acceleration},$$

the time rate of change of linear momentum of an arbitrary portion of material is equal to the resultant force acting on it. Thus,

$$\text{total force} = \begin{matrix} \text{body} \\ \text{forces} \end{matrix} + \begin{matrix} \text{contact} \\ \text{forces} \end{matrix} = \begin{matrix} \text{rate of change in} \\ \text{linear momentum,} \end{matrix}$$

$$\Rightarrow \iiint_{V_t} \underline{f} dV + \iint_S \underline{T} dS = \frac{D}{Dt} \iiint_{V_t} \rho \underline{\dot{x}} dV.$$

$$\text{i.e.,} \quad \iiint_{V_t} \underline{f} dV + \iint_S \underline{T} dS = \iiint_{V_t} \rho \underline{\ddot{x}} dV. \quad (4.3)$$

To prove

$$\frac{D}{Dt} \iiint_{V_t} \rho \underline{\dot{x}} dV = \iiint_{V_t} \rho \underline{\ddot{x}} dV,$$

consider,

$$\begin{aligned} \frac{D}{Dt} \iiint_{V_t} \rho \underline{\dot{x}} dV &= \frac{D}{Dt} \iiint_{V_0} \rho \underline{\dot{x}} J dV_0, & dV &= JdV_0 \\ &= \frac{D}{Dt} \iiint_{V_0} \rho_0 \underline{\dot{x}} dV_0, & \rho J &= \rho_0 \\ &= \iiint_{V_0} \frac{D}{Dt} \{ \rho_0 \underline{\dot{x}} \} dV_0, \\ &= \iiint_{V_0} \rho_0 \underline{\ddot{x}} dV_0, & \frac{D}{Dt} \{ p_0 \} &= 0 \end{aligned}$$

$$\begin{aligned}
&= \iiint_{V_0} \rho \ddot{\underline{x}} J dV_0, & \rho_0 &= \rho J \\
&= \iiint_{V_t} \rho \ddot{\underline{x}} dV. & J dV_0 &= dV
\end{aligned}$$

Therefore,

$$\frac{D}{Dt} \iiint_{V_t} \rho \dot{\underline{x}} dV = \iiint_{V_t} \rho \ddot{\underline{x}} dV,$$

and the conservation of linear momentum (4.3) holds true.

**Note:**

At the moment we can't prove that  $\frac{D}{Dt}\{\rho_0\} = 0$ , but we will later.

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### 4.3.3 Conservation of moment of momentum

Again, based on Newton's second law, the time rate of change of angular momentum of an arbitrary portion of material is equal to the total resultant moment. Thus,

$$\text{total moment} = \text{body moment} + \text{contact moment} = \text{rate of change in angular momentum},$$

$$\iiint_{V_t} \underline{x} \times \underline{f} dV + \iint_S \underline{x} \times \underline{T} dS = \frac{D}{Dt} \iiint_{V_t} \underline{x} \times \rho \dot{\underline{x}} dV,$$

$$\text{i.e.,} \quad \iiint_{V_t} \underline{x} \times \underline{f} dV + \iint_S \underline{x} \times \underline{T} dS = \iiint_{V_t} \underline{x} \times \rho \ddot{\underline{x}} dV. \quad (4.4)$$

To prove

$$\frac{D}{Dt} \iiint_{V_t} \underline{x} \times \rho \dot{\underline{x}} dV = \iiint_{V_t} \underline{x} \times \rho \ddot{\underline{x}} dV,$$

consider,

$$\begin{aligned}
\frac{D}{Dt} \iiint_{V_t} \underline{x} \times \rho \dot{\underline{x}} dV &= \frac{D}{Dt} \iiint_{V_0} \rho \underline{x} \times \dot{\underline{x}} J dV_0, & dV &= J dV_0 \\
&= \frac{D}{Dt} \iiint_{V_0} \rho_0 \underline{x} \times \dot{\underline{x}} dV_0, & \rho J &= \rho_0 \\
&= \iiint_{V_0} \frac{D}{Dt} \{ \rho_0 \underline{x} \times \dot{\underline{x}} \} dV_0, \\
&= \iiint_{V_0} \rho_0 \frac{D}{Dt} \{ \underline{x} \times \dot{\underline{x}} \} dV_0, & \frac{D}{Dt} \{ \rho_0 \} &= 0
\end{aligned}$$

$$\begin{aligned}
&= \iiint_{V_0} \rho_0 \{ \dot{\underline{x}} \times \dot{\underline{x}} + \underline{x} \times \ddot{\underline{x}} \} dV_0, & \text{product rule} \\
&= \iiint_{V_0} \rho_0 \underline{x} \times \ddot{\underline{x}} dV_0 & \dot{\underline{x}} \times \dot{\underline{x}} &= 0 \\
&= \iiint_{V_0} \rho \underline{x} \times \ddot{\underline{x}} J dV_0, & \rho_0 &= \rho J \\
&= \iiint_{V_t} \underline{x} \times \rho \ddot{\underline{x}} dV. & J dV_0 &= dV
\end{aligned}$$

Therefore,

$$\frac{D}{Dt} \iiint_{V_t} \underline{x} \times \rho \dot{\underline{x}} dV = \iiint_{V_t} \underline{x} \times \rho \ddot{\underline{x}} dV,$$

and the conservation of angular momentum (4.4) holds true.

**Note:**

1. These laws are fundamental assumptions of continuum mechanics. They are defined with respect to an inertial frame. Moments are taken about a fixed point.
2. Because the body is deformable and has infinite degrees of freedom in its motion, these laws must hold for every part of the whole body at each time.
3. Note 2 is very important, because it means that if we can express the conservation equations under one set of integrals, then the integrand must be zero, i.e.,

$$\iiint_V \text{integrand } dV = 0 \Rightarrow \text{integrand} = 0.$$

□

## Summary

In this lecture, we ...

- examined the idea of stress
- introduced the Law's of Motion

## Coming up

In the next lecture, we will ...

- express the Law's of Motion in a more useful representation
- introduce principal directions and stresses

**Homework Exercise 4.2:**

1. Sketch three separate general surfaces whose normals are in the  $x$ ,  $y$  and  $z$ -directions respectively, and sketch the location and direction of all relevant stress components of  $\sigma_{ij}$  for each surface.
2. Using Green's theorem, namely

$$\iint_S \phi n_i dS = \iiint_B \frac{\partial \phi}{\partial x^i} dV,$$

show that the conservation of linear momentum (4.3) can be expressed as

$$\iiint_B \left( \frac{\partial \sigma_{ji}}{\partial x^j} + f_i - \rho \ddot{x}^i \right) dV = 0.$$