MATH312

Autumn 2008

Aims

In this lecture, we will ...

- introduce the ideas behind continuum mechanics
- introduce the Einstein summation convention

1 Introduction

MATH312 is about the study of continuum mechanics.

In particular, this course has three components:

1. **Tensors:** Generalized vectors used to describe equations in co-ordinate free notation.
2. **Foundation of continuum mechanics:** Basic results and fundamental equations.
3. **Application of continuum mechanics:** Examining problems in elasticity and at the nano-scale.

But before we start looking at tensors, we will have a look at the ideas behind continuum mechanics.

Question?

What is continuum mechanics? ✠

Answer:

– Mechanics is the study of the motion of bodies of matter and the forces that cause motion.

– Continuum mechanics is the branch of mechanics concerned with continuous bodies of matter.

– For example, the stresses in solids, liquids and gases, and the deformation or flow of these materials.

□

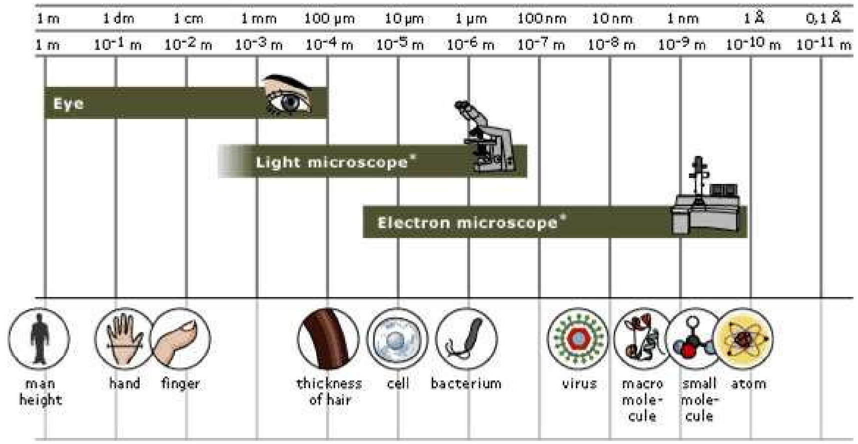
How continuous is continuous?

In the real world, bodies of matter contain particles discretely distributed throughout their body. This means that in reality no media is continuous.

However, as the particles are very small in comparison to the size of the entire body, then it is *reasonable* to assume that matter is distributed continuously – provided the bodies of matter we are considering are on a much larger scale than the atomic scale.

Such assumptions have to be made when mathematically modelling phenomena in the real world, on a realistic scale.

For example, the atoms that make up a pen are infinitesimally small in comparison to the size of the pen.



Assuming atoms are distributed continuously throughout the entire material means the properties of a material, such as density and temperature, are continuous functions of position.

This means that at all points in the body, all material properties exist – allowing us to use what we already know about calculus to model the body's behaviour.

In other words, continuum mechanics disregards the molecular structure of the body of matter, and instead assumes that the body is continuous without any gaps or empty spaces.

Such a hypothetical material is called a *continuous medium* or *continuum.*

Continuum theories, such as elasticity and plasticity, are based on the assumption of continuous materials and lead to quantitative predictions that agree closely with real-life experience over a wide range of conditions.

These theories, together with the simplified engineering continuum theories of beams, plates and shells, are usually adequate for the analysis of the stress and deformation of materials in most practical problems of interest.

The first attempt to mathematically model the behaviour of a material using continuum mechanics usually involves three main assumptions, namely

1. **Continuity:** A material is *continuous* if it completely fills the space that it occupies, leaving no holes or empty space, and its properties are describable by continuous functions.

2. **Homogeneity:** A material is *homogeneous* if it has identical properties at all points in the material.

3. **Isotropy:** A material is *isotropic* with respect to a certain property if the property is identical in all directions.

It is important to understand that the three assumptions of continuity, homogeneity and isotropy are completely independent.

Exercise 1.1:

Give an example of materials with the following properties.

1. Continuous, homogeneous and isotropic.
2. Continuous, inhomogeneous and isotropic.
3. Continuous, homogeneous and anisotropic.
4. Discontinuous, homogeneous and isotropic.
5. Discontinuous, inhomogeneous and anisotropic.

✠

Part 1: Tensors

2 Tensors and Vectors

2.1 Introduction

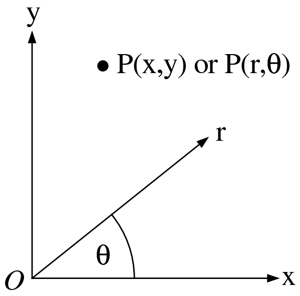
For a physical law to realistically describe the physical world it must be independent of the position and orientation of all observers.

In other words, if two scientists using different coordinate systems observe the same physical event, then it must be possible to state a physical law governing the event in such a way that if the law is true for one observer, then it is also true for the other.

For this reason, physical laws can be written as *vector equations* or *tensor equations*.

Both vector and tensor equations transform from one coordinate system to another in such a way that if a vector or tensor equation holds in one coordinate system, then it holds in any other coordinate system not moving relative to the first one.

This means that the coordinate systems must have the same reference frame, e.g.,



The representation of the point P, in either Cartesian coordinates or polar coordinates, is based from the same origin O.

You should already know about, and have used vectors before; however, we will review some of the ideas behind vectors.

To begin, let us introduce some notation.

2.2 Indicial Notation

Let (x^1, x^2, x^3) denote a coordinate system in $\mathbb{R}^3$. For example:

- in the usual rectangular coordinate system (x, y, z), $x^1 = x$, $x^2 = y$ and $x^3 = z$.
- in cylindrical polar coordinates (r, θ, z), $x^1 = r$, $x^2 = \theta$ and $x^3 = z$.

Further, let $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$ denote three orthogonal base vectors. Note that the base vectors in Cartesian coordinates are usually denoted by $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.

If there is no confusion about the base vectors, then a vector $\underset{\sim}{A}$ can be expressed in the form

$$\underset{\sim}{A} = (A^1, A^2, A^3),$$

while the base vectors can be explicitly shown in the component form

$$\underset{\sim}{A} = A^1\underset{\sim}{e}_1 + A^2\underset{\sim}{e}_2 + A^3\underset{\sim}{e}_3,$$

where A^1, A^2 and A^3 are the components of vector $\underset{\sim}{A}$, which signify the *size* of $\underset{\sim}{A}$ in the directions $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$, respectively.

The component form of a vector is particularly useful when combined with the *Einstein summation convention*, as general results can be written and manipulated in a concise and elegant manner.

The Einstein summation convention says any index that occurs exactly twice in any *term* denotes a sum over the index. For example, if $i = 1, \ldots, 3$ and $j = 2, \ldots, 5$, then

$$a_i x_i = a_1x_1 + a_2x_2 + a_3x_3, \qquad B_{jj} = B_{22} + B_{33} + B_{44} + B_{55},$$

$$\begin{aligned} k_iC_{ij}z_j &= k_1C_{1j}z_j + k_2C_{2j}z_j + k_3C_{3j}z_j, \\ &= k_1(C_{12}z_2 + C_{13}z_3 + C_{14}z_4 + C_{15}z_5) \\ &\quad +k_2(C_{22}z_2 + C_{23}z_3 + C_{24}z_4 + C_{25}z_5) \\ &\quad +k_3(C_{32}z_2 + C_{33}z_3 + C_{34}z_4 + C_{35}z_5). \end{aligned}$$

Summary

In this lecture, we ...

- introduced the ideas behind continuum mechanics
- introduced the Einstein summation convention

Coming up

In the next lecture, we ...

- consider how to classify tensors and basic operations
- introduce the Kronecker delta symbol

Homework Exercise 2.1:

1. In your own words, explain what continuum mechanics is, giving examples of materials that can be modelled using continuum mechanics.
2. If $i = 1, \ldots, 3$ and $j = 0, \ldots 3$, expand the following tensorial quantities using the Einstein summation convention:
   (a) m_{ii},
   (b) $b_j X_j$,
   (c) $c + a_i^i$, where c is a constant,
   (d) $d_{ij} E^{ik} D_l^j$.