Review

In the previous lecture, we ...

- examined pure rotation and pure deformation
- introduced polar decomposition

Aims

In this lecture, we will ...

- determine how to find the square root of a matrix
- introduce the compatibility conditions

Note:

We have seen that a deformation can be broken up into a pure deformation and a pure rotation in one of two ways, namely

$$\underset{\sim}{x} = \mathbf{I}\underset{\sim}{X} + \boldsymbol{e}\underset{\sim}{X} + \boldsymbol{\omega}\underset{\sim}{X} \quad \Rightarrow \quad \mathbf{F} = \mathbf{I} + \boldsymbol{e} + \boldsymbol{\omega}, \quad \text{and} \quad \mathbf{F} = \mathbf{RU}.$$

However, there is a difference between them. The first form is only valid for *small* deformations, while the second form is valid for *all* deformations. In other words,

1. In large deformations, pure deformations and pure rotations are combined by matrix multiplication.
2. In small deformations, pure deformations and pure rotations are combined by addition, or superposition.

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3.5.4 The square root of a matrix

The square root of a *square* matrix $\mathbf{A}$ can be found using the *eigen decomposition*, namely

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where $\mathbf{D}$ is a diagonal matrix constructed from the eigenvalues of $\mathbf{A}$ and $\mathbf{P}$ is a matrix made of the corresponding unit eigenvectors of $\mathbf{A}$.

In this case, if

$$\mathbf{x} = \mathbf{AX} \quad \Rightarrow \quad \mathbf{x} = \mathbf{PDP}^{-1}\mathbf{X} \quad \Rightarrow \mathbf{P}^{-1}\mathbf{x} = \mathbf{DP}^{-1}\mathbf{X},$$

i.e.,

$$\bar{\mathbf{x}} = \mathbf{D}\bar{\mathbf{X}},$$

where $\bar{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{x}$ and $\bar{\mathbf{X}} = \mathbf{P}^{-1}\mathbf{X}$.

Thus, since the same linear transformation $\mathbf{P}^{-1}$ is being applied to both $\mathbf{x}$ and $\mathbf{X}$, then solving the original system is equivalent to solving the transformed system.

Therefore, as we have

$$\mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2},$$

then if we can find the diagonalized matrix $\mathbf{D}$ corresponding to $\mathbf{F}^T\mathbf{F}$, then we can perform the square root as

$$\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}^{1/2} = \begin{pmatrix} \sqrt{d_1} & 0 & 0 \\ 0 & \sqrt{d_2} & 0 \\ 0 & 0 & \sqrt{d_3} \end{pmatrix}.$$

Example 3.4:

Given the deformation

$$x^1 = X^1 + 2X^2, \quad x^2 = X^2, \quad x^3 = X^3,$$

find the right stretch tensor $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$.

Answer:

We first need to find the deformation matrix $\mathbf{F}$, i.e.,

$$\mathbf{F} = \left(\frac{\partial x^i}{\partial X^j}\right) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then,

$$\mathbf{F}^T\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, we need to find the eigenvalues and eigenvectors of $\mathbf{F}^T\mathbf{F}$. Thus, consider

$$|\mathbf{F}^T\mathbf{F} - \lambda\mathbf{I}| = 0,$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0,$$

$$\Rightarrow \quad (1-\lambda)(\lambda^2 - 6\lambda + 1) = 0,$$

$$\text{i.e.,} \quad \lambda = 1, 3+\sqrt{8}, 3-\sqrt{8}.$$

This means that $\mathbf{F}^T\mathbf{F}$ is *equivalent* to

$$\mathbf{D} = \begin{pmatrix} 3+\sqrt{8} & 0 & 0 \\ 0 & 3-\sqrt{8} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and corresponds to the stretches in the principle directions.

Now, as we want $\sqrt{\mathbf{F}^T\mathbf{F}}$, then we calculate $\sqrt{\mathbf{D}}$, i.e.,

$$\sqrt{\mathbf{D}} = \begin{pmatrix} 3+\sqrt{8} & 0 & 0 \\ 0 & 3-\sqrt{8} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{1/2} = \begin{pmatrix} \sqrt{3+\sqrt{8}} & 0 & 0 \\ 0 & \sqrt{3-\sqrt{8}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, to find the corresponding value of $\sqrt{\mathbf{F}^T\mathbf{F}}$, we first need to find the matrix $\mathbf{P}$ corresponding to the eigenvectors. Thus,

1. $\lambda = 3+\sqrt{8}$: $(\mathbf{F}^T\mathbf{F} - (3+\sqrt{8})\cdot\mathbf{I})\underset{\sim}{x} = \underset{\sim}{0}$

$$\Rightarrow \quad \begin{pmatrix} -2-\sqrt{8} & 2 & 0 \\ 2 & 2-\sqrt{8} & 0 \\ 0 & 0 & -2-\sqrt{8} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underset{\sim}{0},$$

$$\Rightarrow \quad x_1 = t, x_2 = (1+\sqrt{2})t, x_3 = 0.$$

2. $\lambda = 3-\sqrt{8}$: $(\mathbf{F}^T\mathbf{F} - (3-\sqrt{8})\cdot\mathbf{I})\underset{\sim}{x} = \underset{\sim}{0}$

$$\Rightarrow \quad \begin{pmatrix} -2+\sqrt{8} & 2 & 0 \\ 2 & 2+\sqrt{8} & 0 \\ 0 & 0 & -2+\sqrt{8} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underset{\sim}{0},$$

$$\Rightarrow \quad x_1 = t, x_2 = (1-\sqrt{2})t, x_3 = 0.$$

3. $\lambda = 1$: $(\mathbf{F}^T\mathbf{F} - 1\cdot\mathbf{I})\underset{\sim}{x} = \underset{\sim}{0}$

$$\Rightarrow \quad \begin{pmatrix} 0 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underset{\sim}{0},$$

$$\Rightarrow \quad x_1 = 0, x_2 = 0, x_3 = t.$$

Therefore, the *unit* eigenvectors are:

$$\lambda = 3+\sqrt{8}: \quad \underset{\sim}{x} = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix}1\\1+\sqrt{2}\\0\end{pmatrix} = \begin{pmatrix}0.3827\\0.9238\\0\end{pmatrix},$$

$$\lambda = 3-\sqrt{8}: \quad \underset{\sim}{x} = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix}1\\1-\sqrt{2}\\0\end{pmatrix} = \begin{pmatrix}0.9238\\-0.3827\\0\end{pmatrix},$$

$$\lambda = 1: \quad \underset{\sim}{x} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Hence, $\mathbf{P}$ is given by

$$\mathbf{P} = \mathbf{P}^{-1} = \begin{pmatrix}0.3827 & 0.9238 & 0\\0.9238 & -0.3827 & 0\\0 & 0 & 1\end{pmatrix}.$$

Thus, finally, $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$ is determine by

$\mathbf{U} = \mathbf{P}\,\sqrt{D}\,\mathbf{P}^{-1}$,

$$= \begin{pmatrix}0.3827 & 0.9238 & 0\\0.9238 & -0.3827 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}2.414 & 0 & 0\\0 & 0.4142 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0.3827 & 0.9238 & 0\\0.9238 & -0.3827 & 0\\0 & 0 & 1\end{pmatrix},$$

i.e.,

$$\mathbf{U} = \begin{pmatrix}0.7070 & 0.7070 & 0\\0.7070 & 2.121 & 0\\0 & 0 & 1\end{pmatrix} \quad\Rightarrow\quad \mathbf{U}^2 = \begin{pmatrix}0.9997 & 1.9994 & 0\\1.9994 & 4.9985 & 0\\0 & 0 & 1\end{pmatrix} \approx \mathbf{F}^T\mathbf{F}.$$

□

Exercise 3.9:

Consider

$$\mathbf{P} = \begin{pmatrix}a & b & 0\\b & -a & 0\\0 & 0 & 1\end{pmatrix},$$

where a and b are constants such that $a^2+b^2=1$. Show that $\mathbf{P}^{-1} = \mathbf{P}$.

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Answer:

□

3.6 Compatibility equations

Recall, that the Lagrangian strain tensor is given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u^m_{,j}).$$

For small displacements, the Lagrangian strain tensor is replaced by the infinitesimal strain tensor ϵ_{ij}, which is symmetric, and is given by

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right),$$

so there are only six independent components of the strain tensor in terms of displacements.

When the displacement functions u_i are given, the six strain components can always be determined in any region where the partial derivatives $u_{i,j}$ exist.

However, when the reverse process is considered, namely when the strain components ϵ_{ij} are given, then mathematical difficulties arise.

In this case, the equations $\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ form a system of six differential equations for the determination of the three displacements u_i.

Thus, it is clear that specifying ϵ_{ij} does not determine the displacements u_i uniquely, as the strain components only characterize the pure deformation of the medium in the neighborhood of $\underset{\sim}{x}$, while the displacements u_i may involve rigid body rotations that do not affect ϵ_{ij}.

Therefore, the solution to the system cannot be unique unless the components of u_i are specified at some point in the medium.

However, if this is done, then there is still a problem - we have six differential equations to determine three functions. In general, you can not expect that there will be a solution for an arbitrary choice of ϵ_{ij}.

We seek further conditions that must be imposed on the functions ϵ_{ij} if the system is to possess a unique solution u_i.

These conditions are called *compatibility conditions*. Here, we only state the conditions as necessary conditions for the existence of single-valued displacements.

In particular, if $\underset{\sim}{u}$ is single-valued and possesses continuous third partial derivatives, then the compatibility conditions are given by

$$\epsilon_{ij,kl} + \epsilon_{kl,ij} - \epsilon_{jl,ik} - \epsilon_{ik,jl} = 0,$$

where

$$\begin{aligned}
\epsilon_{ij,kl} &= \tfrac{1}{2}(u_{i,jkl} + u_{j,ikl}),\\
\epsilon_{kl,ij} &= \tfrac{1}{2}(u_{k,lij} + u_{l,kij}),\\
\epsilon_{jl,ik} &= \tfrac{1}{2}(u_{j,lik} + u_{l,jik}),\\
\epsilon_{ik,jl} &= \tfrac{1}{2}(u_{i,kjl} + u_{k,ijl}).
\end{aligned}$$

The compatibility conditions contain eighty one equations, but only six independent ones, which can be written as

$$\varepsilon_{pks}[\epsilon_{sj,ik} - \epsilon_{si,jk}] = 0. \tag{3.8}$$

The full set of six equations makes the problem very complicated. In practice, therefore, most three-dimensional problems are treated in terms of displacements instead of strains. This satisfies the requirements of compatibility automatically.

However, in two-dimensional problems, all except one of the compatibility equations degenerate to identities, so that a formulation in terms of strains (or stress) is more practical.

A more "physical" way of thinking about compatibility is to state that the separate particles of the body must deform in such a way that they fit together after deformation.

Exercise 3.10:

If $i = s = 1$ and $j = k = 2$, determine the single non-zero compatibility condition from

$$\varepsilon_{pks}[\epsilon_{sj,ik} - \epsilon_{si,jk}] = 0.$$

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Answer:

□

Summary

In this lecture, we ...

- determined how to find the square root of a matrix
- introduced the compatibility conditions

Coming up

In the next lecture, we will ...

- introduce external and internal forces
- introduce traction

Homework Exercise 3.5:

1. Consider the deformation

$$x^1 = 2X^1 - X^2, \quad x^2 = 5X^1 + 3X^2, \quad x^3 = 4X^3.$$

   Find $\mathbf{U}$ and $\mathbf{V}$.

2. Determine the six distinct compatibility equations in full.