Review

In the previous lecture, we ...

- introduced the Lagrangian and Eulerian strain tensors
- considered the concept of incompressible materials

Aims

In this lecture, we will ...

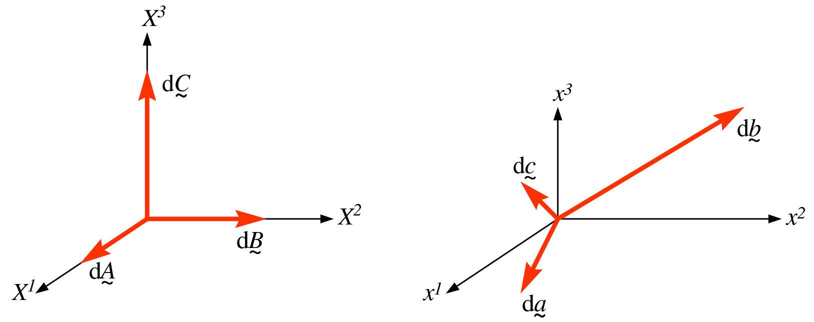
- examine pure rotation and pure deformation
- introduce polar decomposition

3.4.1 Incompressible in terms of deformation

We can also examine the idea of an incompressible deformation by considering the deformation of the three line elements

$$\mathrm{d}\underset{\sim}{A} = (\mathrm{d}X^1, 0, 0), \quad \mathrm{d}\underset{\sim}{B} = (0, \mathrm{d}X^2, 0), \quad \mathrm{d}\underset{\sim}{C} = (0, 0, \mathrm{d}X^3).$$

After undergoing some deformation, these vectors become $\mathrm{d}\underset{\sim}{a}$, $\mathrm{d}\underset{\sim}{b}$ and $\mathrm{d}\underset{\sim}{c}$ respectively.



In particular, we know

$$\mathrm{d}\underset{\sim}{a} = (\mathrm{d}x_a^1, \mathrm{d}x_a^2, \mathrm{d}x_a^3), \quad \mathrm{d}\underset{\sim}{b} = (\mathrm{d}x_b^1, \mathrm{d}x_b^2, \mathrm{d}x_b^3), \quad \mathrm{d}\underset{\sim}{c} = (\mathrm{d}x_c^1, \mathrm{d}x_c^2, \mathrm{d}x_c^3),$$

and are related to the initial vectors by

$$\begin{aligned}
\mathrm{d}x_a^i &= \frac{\partial x_a^i}{\partial X_A^j}\mathrm{d}X_A^j = \frac{\partial x_a^i}{\partial X^1}\mathrm{d}X^1, \\
\mathrm{d}x_b^i &= \frac{\partial x_b^i}{\partial X_B^j}\mathrm{d}X_B^j = \frac{\partial x_b^i}{\partial X^2}\mathrm{d}X^2, \\
\mathrm{d}x_c^i &= \frac{\partial x_c^i}{\partial X_C^j}\mathrm{d}X_C^j = \frac{\partial x_c^i}{\partial X^3}\mathrm{d}X^3,
\end{aligned}$$

where the subscripts denote which vector we are considering.

To find the initial volume, we calculate the scalar triple product

$$\mathrm{d}V_0 = \mathrm{d}\underset{\sim}{A} \times \mathrm{d}\underset{\sim}{B} \cdot \mathrm{d}\underset{\sim}{C} = \begin{vmatrix} \mathrm{d}X^1 & 0 & 0 \\ 0 & \mathrm{d}X^2 & 0 \\ 0 & 0 & \mathrm{d}X^3 \end{vmatrix} = \mathrm{d}X^1\mathrm{d}X^2\mathrm{d}X^3.$$

In comparison, the volume of the deformed region is given by

$$\begin{aligned}
\mathrm{d}V = \mathrm{d}\underset{\sim}{a} \times \mathrm{d}\underset{\sim}{b} \cdot \mathrm{d}\underset{\sim}{c} &= \begin{vmatrix} \frac{\partial x^1}{\partial X^1} & \frac{\partial x^1}{\partial X^2} & \frac{\partial x^1}{\partial X^3} \\ \frac{\partial x^2}{\partial X^1} & \frac{\partial x^2}{\partial X^2} & \frac{\partial x^2}{\partial X^3} \\ \frac{\partial x^3}{\partial X^1} & \frac{\partial x^3}{\partial X^2} & \frac{\partial x^3}{\partial X^3} \end{vmatrix} \mathrm{d}X^1\mathrm{d}X^2\mathrm{d}X^3, \\
&= \left|\frac{\partial x^i}{\partial X^j}\right| \mathrm{d}X^1\mathrm{d}X^2\mathrm{d}X^3, \\
&= J\mathrm{d}V_0.
\end{aligned}$$

Therefore,

$$J = \frac{\mathrm{d}V}{\mathrm{d}V_0},$$

which is the ratio of current deformed volume to the original volume of the material element.

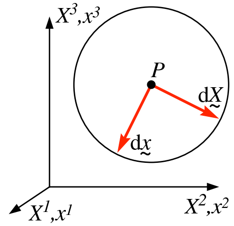
Note:

When $J = 1$, the material is incompressible.

3.5 Deformation

3.5.1 Pure rotation

Consider a pure rotation about a point P in a small region of material. Each line element $\mathrm{d}\underset{\sim}{X}$ through P undergoes only an orientation change, so there is no length change.



Thus,

$$|\mathrm{d}\underset{\sim}{X}| = |\mathrm{d}\underset{\sim}{x}|.$$

Further, the deformation gradient $\mathbf{F}$ is simply equal to the rotation matrix $\mathbf{R}$.

Therefore,

$$\mathrm{d}\underset{\sim}{x} = \mathbf{F}\,\mathrm{d}\underset{\sim}{X} = \mathbf{R}\,\mathrm{d}\underset{\sim}{X}.$$

And, as there is no length change

$$|\mathrm{d}\underset{\sim}{x}|^2 = \mathrm{d}\underset{\sim}{x}\cdot\mathrm{d}\underset{\sim}{x} = \mathrm{d}\underset{\sim}{X}\cdot\mathrm{d}\underset{\sim}{X} = \mathrm{d}\underset{\sim}{X}^{T}\mathrm{d}\underset{\sim}{X},$$

but

$$\mathrm{d}\underset{\sim}{x}\cdot\mathrm{d}\underset{\sim}{x} = (\mathbf{R}\,\mathrm{d}\underset{\sim}{X})\cdot(\mathbf{R}\,\mathrm{d}\underset{\sim}{X}) = \mathrm{d}\underset{\sim}{X}^{T}\mathbf{R}^{T}\mathbf{R}\,\mathrm{d}\underset{\sim}{X}.$$

Thus,

$$\mathbf{R}^{T}\mathbf{R} = \mathbf{I} \quad \Rightarrow \quad \mathbf{R}^{T} = \mathbf{R}^{-1},$$

which means that $\mathbf{R}$ is an orthogonal tensor.

Note:

If we can show that $\mathbf{F}^{T}\mathbf{F} = \mathbf{I}$, then $\mathbf{F} = \mathbf{R}$. Further, it can also be shown that $\mathrm{d}\underset{\sim}{x} = \mathbf{R}\,\mathrm{d}\underset{\sim}{X}$ preserves the angles between the lines.

Now, the change in volume of the region is given by

$$J = \left|\frac{\partial x^i}{\partial X^j}\right| = |\mathbf{R}^i_j|,$$

and

$$|\mathbf{R}^t\mathbf{R}| = |\mathbf{R}^t||\mathbf{R}| = |\mathbf{R}|^2 = |\mathbf{I}| = 1,$$

so that $J = 1$. Therefore, a pure rotation is incompressible.

Exercise 3.7:

For a particular deformation, the deformation gradient is given by

$$\mathbf{F} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

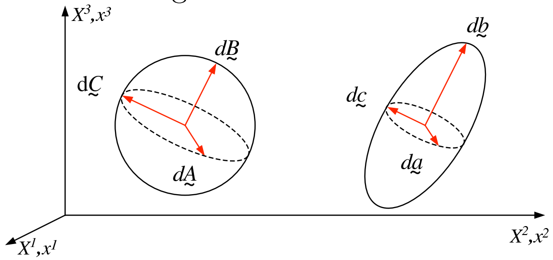
Is this deformation only a rotation?

Answer:

□

3.5.2 Pure deformation

Consider a pure deformation of a point P in a small region of material. Each line element $\mathrm{d}\underset{\sim}{X}$ through P undergoes only a length change, so there is no change in orientation.



In this case, the deformation gradient $\mathbf{F} = \mathbf{U}$, i.e.,

$$\mathrm{d}\underset{\sim}{x} = \mathbf{U}\,\mathrm{d}\underset{\sim}{X},$$

where $\mathbf{U}^i_j = \mathbf{U}^j_i$.

Now, as $\mathrm{d}\underset{\sim}{a} \parallel \mathrm{d}\underset{\sim}{A}$, $\mathrm{d}\underset{\sim}{b} \parallel \mathrm{d}\underset{\sim}{B}$, and $\mathrm{d}\underset{\sim}{c} \parallel \mathrm{d}\underset{\sim}{C}$, then

$$\mathrm{d}x^i = \mathbf{U}^i_j\,\mathrm{d}X^j = \lambda_{\underline{i}}\,\mathrm{d}X^{\underline{i}}, \qquad \lambda_i = \text{ scalar},$$

or,

$$(\mathbf{U}^i_j - \lambda_{\underline{i}}\,\delta^{\underline{i}}_j)\,\mathrm{d}X^j = 0. \tag{3.7}$$

There are at most three values of λ_i and at least three distinct vectors $\mathrm{d}\underset{\sim}{X}$ that satisfy (3.7).

If the three values of λ_i are distinct, then there are only three mutually perpendicular vectors satisfying (3.7), and

$$\mathrm{d}\underset{\sim}{a} = \lambda_a \mathrm{d}\underset{\sim}{A}, \quad \mathrm{d}\underset{\sim}{b} = \lambda_b \mathrm{d}\underset{\sim}{B}, \quad \mathrm{d}\underset{\sim}{c} = \lambda_c \mathrm{d}\underset{\sim}{C}.$$

3.5.3 Polar decomposition

Ignoring translations, any deformation can be resolved into a pure rotation plus a pure deformation, or vice-versa.

To resolve a general deformation into a pure rotation and a pure deformation, $\mathbf{F}$ must be non-singular, in which case there exists a unique decomposition into either of the products:

$$\mathbf{F} = \mathbf{R}\,\mathbf{U},$$

or

$$\mathbf{F} = \mathbf{V}\mathbf{R},$$

where $\mathbf{R}$ is an orthogonal tensor, while $\mathbf{U}$ and $\mathbf{V}$ are both positive-definite symmetric tensors.

Hence, a general deformation can be thought of in two ways:

(1). A pure deformation followed by a pure rotation:

$$\mathbf{F} = \mathbf{R}\,\mathbf{U}$$

(2). A pure rotation followed by a pure deformation:

$$\mathbf{F} = \mathbf{V}\mathbf{R}$$

The same rotation occurs in both forms, and while $\mathbf{U}$ and $\mathbf{V}$ give the same principal stretches, the principal directions of stretch differ by the rotation $\mathbf{R}$.

Note:

$\mathbf{U}$ is called the right stretch tensor, while $\mathbf{V}$ is called the left stretch tensor.

♪

Since $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$, then

$$\mathbf{R}\mathbf{U}\mathbf{R}^T = \mathbf{V}\mathbf{R}\mathbf{R}^T = \mathbf{V}\mathbf{I} = \mathbf{V},$$

i.e.,
$$\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T,$$

and similarly, $\mathbf{U} = \mathbf{R}^T\mathbf{V}\mathbf{R}$. Thus, we can calculate $\mathbf{U}$ or $\mathbf{V}$ given the other one and $\mathbf{R}$.

Note:

As $\mathbf{U}$ and $\mathbf{V}$ are symmetric, then

$$\mathbf{F}^T\mathbf{F} = (\mathbf{R}\mathbf{U})^T(\mathbf{R}\mathbf{U}) = \mathbf{U}^T\mathbf{R}^T\mathbf{R}\mathbf{U} = \mathbf{U}^T\mathbf{U} = \mathbf{U}^2 \quad \Rightarrow \quad \mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2},$$
$$\mathbf{F}\mathbf{F}^T = (\mathbf{V}\mathbf{R})(\mathbf{V}\mathbf{R})^T = \mathbf{V}\mathbf{R}\mathbf{R}^T\mathbf{V}^T = \mathbf{V}\mathbf{V}^T = \mathbf{V}^2 \quad \Rightarrow \quad \mathbf{V} = (\mathbf{F}\mathbf{F}^T)^{1/2}.$$

Thus, we need to know how to square root a matrix to find $\mathbf{U}$ and $\mathbf{V}$.

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Exercise 3.8:

Given

$$x^1 = X^1, \quad x^2 = -3X^3, \quad x^3 = 2X^2,$$

Find

1. the deformation gradient $\mathbf{F}$,
2. the right stretch tensor $\mathbf{U}$,
3. the rotation tensor $\mathbf{R}$, and
4. the left stretch tensor $\mathbf{V}$.

✠

Answer:

□

Summary

In this lecture, we ...

- examine pure rotation and pure deformation
- introduce polar decomposition

Coming up

In the next lecture, we will ...

- determine how to find the square root of a matrix
- introduce the compatibility conditions

Homework Exercise 3.4:

1. Consider the deformation

$$x^1 = 5X^2, \quad x^2 = -2X^1, \quad x^3 = X^3.$$

   Find
   (a) the deformation gradient $\mathbf{F}$,
   (b) the right stretch tensor $\mathbf{U}$,
   (c) the rotation tensor $\mathbf{R}$, and
   (d) the left stretch tensor $\mathbf{V}$.

2. For what values of α is the deformation

$$x^1 = 12\alpha X^1 + 3X^2, \quad x^2 = -3X^1 + 12\alpha X^2, \quad x^3 = X^3,$$

   a rotation only. What do these values imply?