Review

In the previous lecture, we …

- considered the transformation of an arbitrary element
- determined physical meanings of the strain and rotation matrices

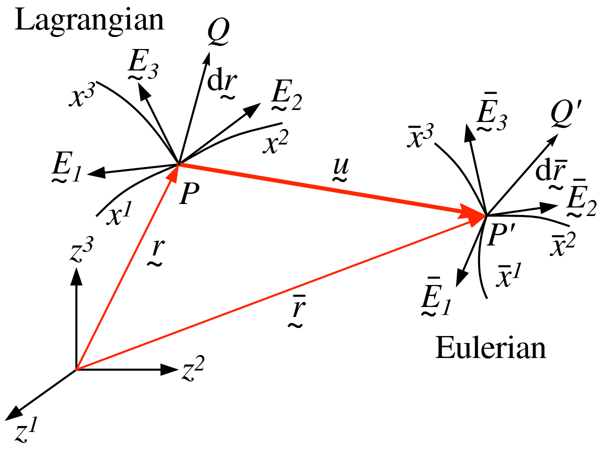
Aims

In this lecture, we will …

- introduce the Lagrangian and Eulerian strain tensors
- consider the concept of incompressible materials

3.3 Lagrangian strain tensor

Consider strain in the generalized coordinates illustrated below:



(z^1, z^2, z^3) Cartesian
(x^1, x^2, x^3) unstrained
$(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ strained

P represents general point in unstrained continuum.

P deforms to P'.

$(\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3)$, $(\bar{\underset{\sim}{E}}_1, \bar{\underset{\sim}{E}}_2, \bar{\underset{\sim}{E}}_3)$ are tangential basis vectors.

In the Lagrangian system of unbarred coordinates, we have the basis vectors $\underset{\sim}{E}_i = \frac{\partial \underset{\sim}{r}}{\partial x^i}$, which gives the metric tensor $g_{ij} = \underset{\sim}{E}_i \cdot \underset{\sim}{E}_j$.

Similarly, in the Eulerian system of barred coordinates, we have the basis vectors $\bar{\underset{\sim}{E}}_i = \frac{\partial \bar{\underset{\sim}{r}}}{\partial \bar{x}^i}$, which gives the metric tensor $\bar{G}_{ij} = \bar{\underset{\sim}{E}}_i \cdot \bar{\underset{\sim}{E}}_j$.

We assume an element of arc length squared $\mathrm{d}s^2$ in the unstrained state is deformed to the element of arc length squared $\mathrm{d}\bar{s}^2$ in the strained state.

An element of arc length squared is invariant under coordinate transformation, and thus can be expressed in terms of barred or unbarred coordinates.

Thus, in the Lagrangian system, we have

$$\mathrm{d}s^2 = \mathrm{d}\underset{\sim}{r} \cdot \mathrm{d}\underset{\sim}{r} = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \bar{g}_{ij}\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j,$$

where $\bar{g}_{ij} = g_{mn}\dfrac{\partial x^m}{\partial \bar{x}^i}\dfrac{\partial x^n}{\partial \bar{x}^j}$.

Similarly, in the Eulerian system, we have

$$\mathrm{d}\bar{s}^2 = \mathrm{d}\bar{\underset{\sim}{r}} \cdot \mathrm{d}\bar{\underset{\sim}{r}} = \bar{G}_{ij}\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j = G_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

where $G_{ij} = \bar{G}_{mn}\dfrac{\partial \bar{x}^m}{\partial x^i}\dfrac{\partial \bar{x}^n}{\partial x^j}$.

If we consider the difference $\mathrm{d}\bar{s}^2 - \mathrm{d}s^2$ in Lagrangian coordinates, then we find

$$\mathrm{d}\bar{s}^2 - \mathrm{d}s^2 = (G_{ij} - g_{ij})\mathrm{d}x^i\mathrm{d}x^j = 2e_{ij}\mathrm{d}x^i\mathrm{d}x^j.$$

Thus,

$$e_{ij} = \frac{1}{2}(G_{ij} - g_{ij})$$

is called the *Green strain tensor* or *Lagrangian strain tensor*.

Similarly, if we consider the difference $\mathrm{d}\bar{s}^2 - \mathrm{d}s^2$ in Eulerian coordinates, then we find

$$\mathrm{d}\bar{s}^2 - \mathrm{d}s^2 = (\bar{G}_{ij} - \bar{g}_{ij})\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j = 2\bar{e}_{ij}\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j,$$

where

$$\bar{e}_{ij} = \frac{1}{2}(\bar{G}_{ij} - \bar{g}_{ij})$$

is called the *Almansi strain tensor* or *Eulerian strain tensor*.

Note:

It is more common to use the Lagrangian strain tensor.

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Now, if $\utilde{u} = u^i \utilde{E}_i$ is the displacement from P and P', then

$$\utilde{r} + \utilde{u} = \utilde{\bar{r}} \qquad \Rightarrow \qquad \mathrm{d}\utilde{r} + \mathrm{d}\utilde{u} = \mathrm{d}\utilde{\bar{r}}.$$

Further, $\mathrm{d}\utilde{r} = \mathrm{d}x^i \utilde{E}_i$, and assuming

$$\mathrm{d}\utilde{u} = u^i{}_{,k}\mathrm{d}x^k \utilde{E}_i, \tag{3.6}$$

then

$$\mathrm{d}\utilde{\bar{r}} = (\mathrm{d}x^i + u^i{}_{,k}\mathrm{d}x^k)\utilde{E}_i.$$

Note:

The proof of equation (3.6) is not considered in MATH312, as it requires the introduction of *Christoffel symbols*.

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Having done this, we can now express $\mathrm{d}\bar{s}^2$ in terms of Lagrangian displacements, i.e.,

$$\begin{aligned}\mathrm{d}\bar{s}^2 = \mathrm{d}\utilde{\bar{r}} \cdot \mathrm{d}\utilde{\bar{r}} &= (\mathrm{d}x^i + u^i{}_{,k}\mathrm{d}x^k)\utilde{E}_i \cdot (\mathrm{d}x^j + u^j{}_{,m}\mathrm{d}x^m)\utilde{E}_j,\\ &= (\mathrm{d}x^i\mathrm{d}x^j + u^j{}_{,m}\mathrm{d}x^m\mathrm{d}x^i + u^i{}_{,k}\mathrm{d}x^k\mathrm{d}x^j\\ &\qquad + u^i{}_{,k}u^j{}_{,m}\mathrm{d}x^k\mathrm{d}x^m)g_{ij}.\end{aligned}$$

Thus, from $\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, we find

$$\begin{aligned}\mathrm{d}\bar{s}^2 - \mathrm{d}s^2 &= (u^j{}_{,m}\mathrm{d}x^m\mathrm{d}x^i + u^i{}_{,k}\mathrm{d}x^k\mathrm{d}x^j + u^i{}_{,k}u^j{}_{,m}\mathrm{d}x^k\mathrm{d}x^m)g_{ij},\\ &= u_{i,m}\mathrm{d}x^m\mathrm{d}x^i + u_{j,k}\mathrm{d}x^k\mathrm{d}x^j + u_{j,k}u^j{}_{,m}\mathrm{d}x^k\mathrm{d}x^m,\\ &= (u_{i,j} + u_{j,i} + u_{m,i}u^m{}_{,j})\mathrm{d}x^i\mathrm{d}x^j.\end{aligned}$$

Therefore, the Lagrangian strain tensor can be given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u^m{}_{,j}).$$

Similarly, we can show that in terms of displacement, the Eulerian strain tensor can be expressed as

$$\bar{e}_{ij} = \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i} + \bar{u}_{m,i}\bar{u}^m{}_{,j}).$$

Note:

In linear elasticity, i.e., small deformations, the Lagrangian and Eulerian strain tensors reduce to

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

$$\bar{e}_{ij} = \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i}).$$

These results match with the corresponding small strain deformations given in the previous lecture, i.e., the strain matrix.

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Example 3.3:

Consider the two-dimensional stretching deformation

$$\bar{x}^1 = (1+\lambda)x^1, \qquad \bar{x}^2 = (1+\gamma)x^2, \qquad \bar{x}^3 = x^3,$$

where $\lambda, \gamma > 0$. In this case, the metric tensor becomes

$$g_{ij} = \begin{pmatrix} (1+\lambda)^2 & 0 & 0 \\ 0 & (1+\gamma)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the displacements are

$$u^1 = \bar{x}^1 - x^1 = \lambda x^1, \qquad u^2 = \bar{x}^2 - x^2 = \gamma x^2, \qquad u^3 = 0.$$

However, as the Lagrangian strain tensor is given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u^m_{,j}),$$

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then we require u_1, u_2 and u_3. Hence, as $u_i = g_{ij}u^j$, then we find

$$\begin{aligned} u_1 &= g_{1j}u^j = (1+\lambda)^2 u^1 = \lambda(1+\lambda)^2 x^1, \\ u_2 &= g_{2j}u^j = (1+\gamma)^2 u^2 = \gamma(1+\gamma)^2 x^2, \\ u_3 &= g_{3j}u^j = u^3 = 0. \end{aligned}$$

Thus,

$$\begin{aligned} e_{11} &= \frac{1}{2}(u_{1,1} + u_{1,1} + u_{m,1}u^m_{,1}), \\ &= \frac{1}{2}(2\lambda(1+\lambda)^2 + \lambda(1+\lambda)^2 \times \lambda), \\ &= \lambda(1+\lambda)^2 + \frac{1}{2}\lambda^2(1+\lambda)^2. \end{aligned}$$

$$e_{12} = \frac{1}{2}(u_{1,2} + u_{2,1} + u_{m,1}u^m_{,2}),$$

$$e_{12} = \frac{1}{2}(0 + 0 + u_{1,1}u^1_{,2} + u_{2,1}u^2_{,2} + u_{3,1}u^3_{,2}) = 0,$$

and so forth. Finally, we find

$$e_{ij} = \begin{pmatrix} \lambda(1+\lambda)^2 + \frac{1}{2}\lambda^2(1+\lambda)^2 & 0 & 0 \\ 0 & \gamma(1+\gamma)^2 + \frac{1}{2}\gamma^2(1+\gamma)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we consider linear elasticity, i.e., small displacements $\Rightarrow \lambda, \gamma \ll 1$, then the Lagrangian strain tensor becomes simply

$$e_{ij} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Exercise 3.5:

For a particular deformation, the Eulerian strain tensor is found to be

$$\bar{e}_{ij} = \begin{pmatrix} \lambda + \lambda\gamma & \lambda^2 & 0 \\ \lambda^2 & \gamma - \lambda\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where λ, γ denote the coordinate stretches in the $\bar{x}^1$ and $\bar{x}^2$ directions respectively. What is the resulting Eulerian strain tensor if the assumption of linear elasticity is imposed?

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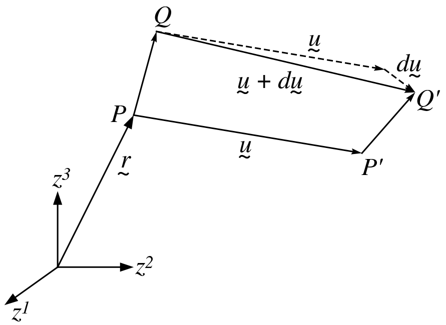
Answer:

□

3.4 Compressible and incompressible material

Consider



P, Q are neighbouring points. P', Q' are the deformed neighbouring points.

Clearly,

$$\overrightarrow{PQ} + \utilde{u} + \mathrm{d}\utilde{u} = \utilde{u} + \overrightarrow{P'Q'},$$

so that if $\overrightarrow{PQ} = \mathrm{d}x^i \utilde{E}_i = a^i \utilde{E}_i$ and $\overrightarrow{P'Q'} = \mathrm{d}\bar{x}^i = A^i \utilde{E}_i$, then using (3.6) we find

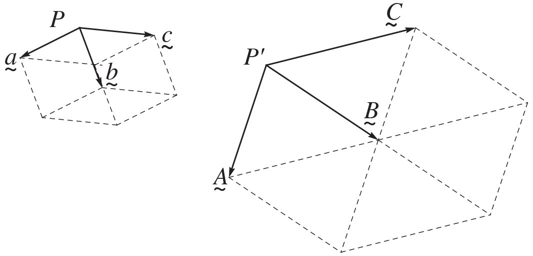
$$\overrightarrow{P'Q'} = \overrightarrow{PQ} + \mathrm{d}\utilde{u} \Rightarrow A^i = a^i + u^i_{,j} a^j.$$

Thus, upon examining the difference $\mathrm{d}\bar{x}^i - \mathrm{d}x^i$, we find

$$A^i - a^i = \delta a^i = u^i_{,j} a^j,$$

where δa^i is an arbitrary small change of the arc length in the x^i-direction.

If we now consider a small parallelepiped element of volume of continuum material whose sides are three independent vectors, i.e.,



Then the question is how does the volume change under strain?

Let ΔV denote the volume of the parallelepiped with sides $\utilde{a}$, $\utilde{b}$, $\utilde{c}$ at P before strain and let $\Delta V'$ denote the volume of the parallelepiped after strain, with sides $\utilde{A}$, $\utilde{B}$, $\utilde{C}$ at P'.

We define the ratio of change in volume due to strain to the original volume as the *dilatation* at P. Thus,

$$\Theta = \frac{\Delta V' - \Delta V}{\Delta V} = \text{dilatation.}$$

Now, we know that the vectors $\utilde{A}$, $\utilde{B}$, $\utilde{C}$ can be expressed in the form

$$\begin{aligned} A^i &= a^i + u^i_{,j} a^j, \\ B^i &= b^i + u^i_{,j} b^j, \\ C^i &= c^i + u^i_{,j} c^j, \end{aligned}$$

where $\utilde{a}$, $\utilde{b}$, $\utilde{c}$ are the arbitrary small vectors at P.

To find the volume element ΔV, we know

$$\Delta V = \utilde{a} \cdot (\utilde{b} \times \utilde{c}) = a^i b^j c^k \varepsilon_{ijk}.$$

Similarly, the volume element $\Delta V'$ is given by

$$\begin{aligned} \Delta V' &= \utilde{A} \cdot (\utilde{B} \times \utilde{C}) = A^i B^j C^k \varepsilon_{ijk}, \\ &= (a^i + u^i_{,m} a^m)(b^j + u^j_{,n} b^n)(c^k + u^k_{,p} c^p) \varepsilon_{ijk}, \end{aligned}$$

so that, after simplification, we find

$$\Theta = \frac{\Delta V' - \Delta V}{\Delta V} = u^r_{,r} = \operatorname{div} \utilde{u}.$$

Therefore, the dilatation is the divergence of the displacement field.

If $\operatorname{div} \underset{\sim}{u} = 0 \Rightarrow \Delta V' = \Delta V$, i.e., there is no change in volume, and the material is said to be *incompressible*.

If $\operatorname{div} \underset{\sim}{u} \neq 0 \Rightarrow \Delta V' \neq \Delta V$, i.e., there is a change in volume, and the material is said to be *compressible*.

Exercise 3.6:

From Example 3.3, we found the displacement field

$$u_1 = \lambda x^1, \quad u_2 = \gamma x^2, \quad u^3 = 0.$$

Under what condition is the material incompressible?

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Answer:

□

Summary

In this lecture, we . . .

- introduce the Lagrangian and Eulerian strain tensors
- consider the concept of incompressible materials

Coming up

In the next lecture, we will . . .

- examine pure rotation and pure deformation
- introduce polar decomposition

Homework Exercise 3.3:

1. Consider the two-dimensional deformation

$$\bar{x}^1 = x^1 + \lambda x^2, \quad \bar{x}^2 = \gamma x^1 + x^2, \quad \bar{x}^3 = x^3,$$

where γ, λ are scalar constants. Find the Lagrangian and Eulerian strain tensors. Under what conditions is this deformation incompressible?

2. Given the above lecture, prove

$$\Theta = \frac{\Delta V' - \Delta V}{\Delta V} = u^r_{,r} = \operatorname{div} \underset{\sim}{u}.$$