Review

In the previous lecture, we ...

- started Part 2: Foundation of continuum mechanics
- introduced Lagrangian and Eulerian description

Aims

In this lecture, we will ...

- consider the transformation of an arbitrary element
- determine physical meanings of the strain and rotation matrices

3.2 Transformation of an arbitrary element

In the Week 7 Assignment, you are asked to consider the deformation of a unit square according to given deformations.

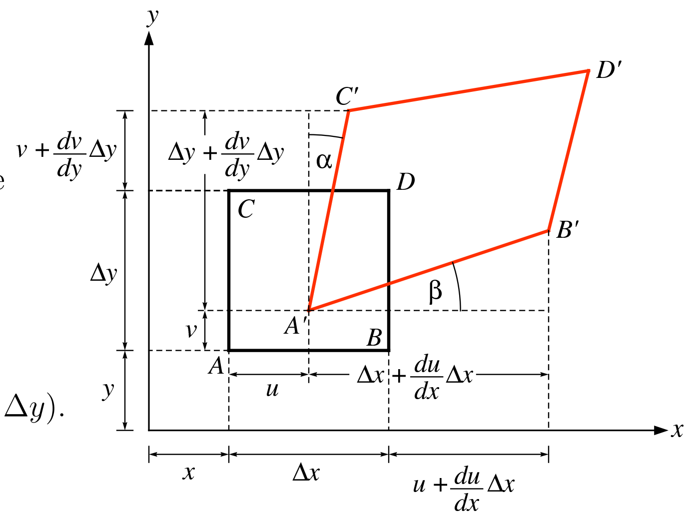
Here, we will consider the general deformation of an arbitrary element, but only in two-dimensions. We will consider the deformation of a rectangular element of material.

While the results derived here can be extended to three (and higher) dimensions, we will only state them here.

For simplicity, we will assume Cartesian coordinates (x, y).

Consider the following rectangular element $ABCD$:

$ABCD$ has the coordinates:
$A(x, y)$,
$B(x + \Delta x, y)$,
$C(x, y + \Delta y)$,
$D(x + \Delta x, y + \Delta y)$.



And, $u = u(x, y)$ and $v = v(x, y)$ are the displacements.

We can assume that the deformation of $ABCD$ can be represented by the matrix equation

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which governs the deformation of each point, and the coefficients b_{ij} are to be determined.

If $u = u(x, y)$ and $v = v(x, y)$ are the horizontal and vertical displacements of point A to A', then

$$u = \bar{x} - x, \qquad v = \bar{y} - y,$$

or

$$\bar{x} = x + u, \qquad \bar{y} = y + v.$$

Thus, the b_{ij}'s must satisfy

$$\begin{pmatrix} x+u \\ y+v \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.1}$$

Next, if we consider the displacement of the point B to B', then the displacement is given by

$$\bar{x} = x + \Delta x + u(x+\Delta x, y), \quad \bar{y} = y + v(x+\Delta x, y),$$

which using Taylor Series gives

$$\bar{x} = x + \Delta x + u + \frac{\partial u}{\partial x}\Delta x + O(\Delta x^2),$$
$$\bar{y} = y + v + \frac{\partial v}{\partial x}\Delta x + O(\Delta x^2).$$

Hence, the b_{ij}'s must also satisfy

$$\begin{pmatrix} x + \Delta x + u + \frac{\partial u}{\partial x}\Delta x \\ y + v + \frac{\partial v}{\partial x}\Delta x \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x+\Delta x \\ y \end{pmatrix}. \tag{3.2}$$

Likewise, if we consider the displacement of C to C', we find

$$\begin{pmatrix} x + u + \frac{\partial u}{\partial y}\Delta y \\ y + \Delta y + v + \frac{\partial v}{\partial y}\Delta y \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x \\ y+\Delta y \end{pmatrix}. \tag{3.3}$$

Finally, the displacement of D to D' gives

$$\begin{pmatrix} x + \Delta x + u + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y \\ y + \Delta y + v + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x+\Delta x \\ y+\Delta y \end{pmatrix}. \tag{3.4}$$

Upon solving the equations (3.1) - (3.4), we find

$$b_{11} = 1 + \frac{\partial u}{\partial x}, \qquad b_{12} = \frac{\partial u}{\partial y},$$
$$b_{21} = \frac{\partial v}{\partial x}, \qquad b_{22} = 1 + \frac{\partial v}{\partial y}.$$

Note:

In general, $b_{12} \neq b_{21}$.

♪

Thus, the transformation/deformation equation becomes

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 1 + \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & 1 + \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

A physical interpretation associated with this deformation is obtain by writing it in the form:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{identity}} \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}}_{\text{strain matrix}} \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}}_{\text{rotation matrix}} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.5}$$

where

$$e_{11} = \frac{\partial u}{\partial x}, \qquad e_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

$$e_{21} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \qquad e_{22} = \frac{\partial v}{\partial y},$$

are the elements of a symmetric matrix called the strain matrix, and

$$\omega_{11} = 0, \qquad \omega_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right),$$

$$\omega_{21} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \qquad \omega_{22} = 0,$$

are the elements of a skew symmetric matrix called the rotation matrix.

Exercise 3.3:

If the displacement field is given by

$$\underset{\sim}{u} = (x^2 + yx)\underset{\sim}{e}_1 + (xy - y^2)\underset{\sim}{e}_2,$$

Find:

1. The strain matrix at the point (1, 2).
2. The rotation matrix at the point (1, 2).

✠

Answer:

□

The question arises as to what do the quantities in the strain and rotation matrices mean physically, i.e., $e_{11}, e_{12}, e_{21}, e_{22}, \omega_{12}$ and ω_{21}.

To answer this, consider the strain in the x-direction associated with the point A, which is given by

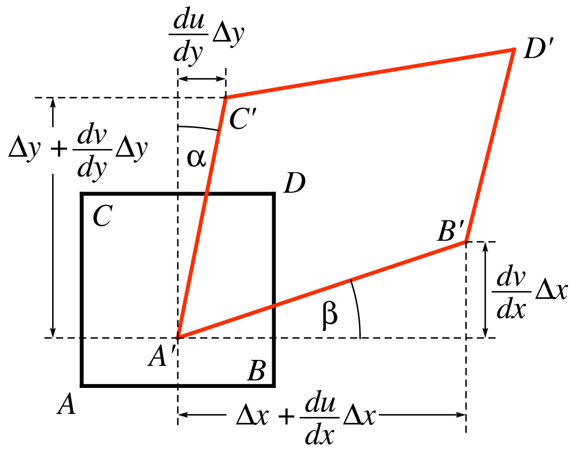
$$\frac{\text{change in length } AB}{\text{original length } AB} = \frac{\Delta x + \frac{\partial u}{\partial x}\Delta x - \Delta x}{\Delta x} = \frac{\partial u}{\partial x} = e_{11},$$

while the strain in the y-direction associated with the point A is given by

$$\frac{\text{change in length } AB}{\text{original length } AB} = \frac{\Delta y + \frac{\partial v}{\partial y}\Delta y - \Delta y}{\Delta y} = \frac{\partial v}{\partial y} = e_{22}.$$

Thus, e_{11} and e_{22} are the strains in the x and y-directions respectively. These quantities are called *normal strains*, because they act in the direction normal to the surface element.

Now, to gain an understanding of e_{12} and e_{21}, we consider the rotation of the line elements AB and AC, i.e.,



Thus, we find

$$\tan\beta = \frac{\frac{\partial v}{\partial x}\Delta x}{\Delta x + \frac{\partial u}{\partial x}\Delta x},$$

$$\tan\alpha = \frac{\frac{\partial u}{\partial y}\Delta y}{\Delta y + \frac{\partial v}{\partial y}\Delta y}.$$

Assuming small displacements u and v, and in turn small derivatives, then the above equations simplify to give

$$\tan\beta \approx \beta = \frac{\partial v}{\partial x}, \qquad \tan\alpha \approx \alpha = \frac{\partial u}{\partial y},$$

noting that as

$$u \ll 1, \quad \text{and} \quad v \ll 1,$$

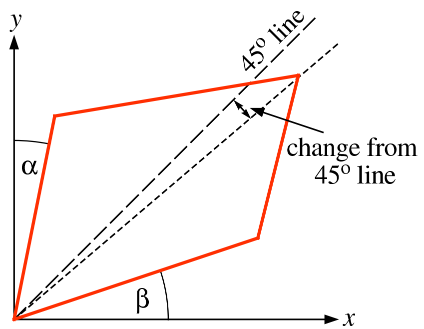
then

$$\frac{\partial u}{\partial x} \ll 1, \quad \text{and} \quad \frac{\partial v}{\partial y} \ll 1.$$

Thus, if we consider the quantity $\alpha + \beta$, then we find

$$\alpha + \beta = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 2e_{12} = 2e_{21}.$$

To determine the physical interpretation, consider the deformation of a small square element that undergoes some shearing as illustrated below:



Then, assuming $\alpha, \beta \ll 1$, we find that

$$\tfrac{1}{2}(\alpha + \beta) = e_{12} = e_{21}$$

represents the change from a 45° angle due to the deformation.

Further, the quantities e_{12} and e_{21} are called *shear strains*, while the quantity

$$\gamma_{12} = 2e_{12},$$

is called the shear angle. (c.f. the quantities e_{11} and e_{22} are called normal strains.)

Note:

In general, $\gamma_{ij} = 2e_{ij}$.

♪

From (3.5), The quantities $\omega_{12} = -\omega_{21}$ correspond to a *rigid body* rotation of the material, and are interpreted as angles associated with the rotation.

Exercise 3.4:

Find the shear angle γ_{21}, if

$$\underset{\sim}{u} = xy\underset{\sim}{e}_1 + y^2\underset{\sim}{e}_2.$$

✠

Answer:

□

Summary

In this lecture, we . . .

- considered the transformation of an arbitrary element
- determined physical meanings of the strain and rotation matrices

Coming up

In the next lecture, we will . . .

- introduce the Lagrangian strain tensor
- introduce Green's deformation tensor

Homework Exercise 3.2:

1. If the displacement field is given by

$$\underset{\sim}{v} = (xy^2 - x^3)\underset{\sim}{e}_1 + (yx^2 + y^3)\underset{\sim}{e}_2,$$

   then find the strain matrix, rotation matrix and the shear angles.

2. Given the above lecture, prove $\frac{1}{2}(\alpha + \beta) = e_{12} = e_{21}$ for small α and β.