Review

In the previous lecture, we …

- introduce scalar and vector invariants
- summarize tensors

Aims

In this lecture, we will …

- start Part 2: Foundation of continuum mechanics
- introduce Lagrangian and Eulerian description

Part 2: Continuum mechanics

3 Strain and deformation

In this section, we will examine the *strain* and *deformation* of continuous medium.

Definition 3.1:

Strain is the deformation of a material caused by stress.

$\square$

This raises the question of what is *stress*?

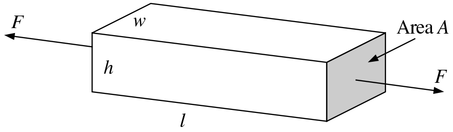
Definition 3.2:

Stress is the force per unit area exerted on a particular surface.

$\square$

Example 3.1:

Consider an elastic rectangular prism of length l, width w and height h, as shown below.



If force F is applied to the prism, then

1. l increases by an amount Δl.
2. w decreases by an amount Δw.
3. h decreases by an amount Δh.

The measurement of the *relative size of change* gives rise to the concept of strain.

In particular, strain is given by

$$\text{strain} = \frac{\text{change in length}}{\text{original length}},$$

and hence, strain is a dimensionless quantity.

Here, in this example, the following three strains arise

$$\frac{\Delta l}{l}, \qquad \frac{\Delta w}{w}, \qquad \frac{\Delta h}{h},$$

in the appropriate directions.

Thus, strain *happens* in a particular direction.

The three strains in this example are caused by a stress.

Stress is defined to be the force per unit area. In particular,

$$\text{stress} = \frac{\text{force}}{\text{area over which force acts}},$$

which implies stress has the dimension of *force per unit area*.

Here, the force F is applied over the area A, so that

$$\text{stress} = \sigma = \frac{F}{A}.$$

Note:

Strain in the three directions is caused by stress in one direction.

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To begin studying strain, we will consider the *kinematics* of continuous medium.

Definition 3.3:

Kinematics describes the deformation of a material without consideration of the forces that cause the deformation.

□

Example 3.2:

If we derived an equation to describe how an elastic material deforms under tension solely due to the material properties, then this is a kinematical equation - not a dynamical equation.

□

3.1 Kinematics of a continuous medium

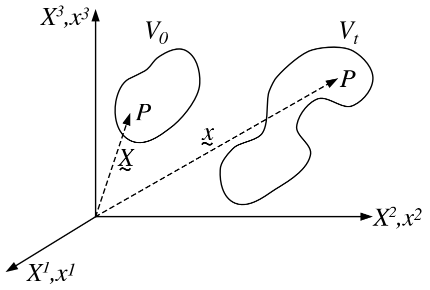
Consider a body of material that occupies a region V_0 at time $t = 0$, and V_t at time t, with material coordinates $\underset{\sim}{X}$ and spatial coordinates $\underset{\sim}{x}$. In general, $V_0 \neq V_t$.

Material description:

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t)$$

Spatial description:

$$\underset{\sim}{X} = \underset{\sim}{X}(\underset{\sim}{x}, t)$$



Note:

The axes X^i and x^i won't lie in the same directions in general - but they must have the same origin.

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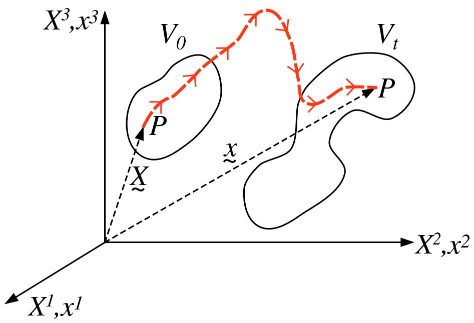
If we consider a point P in V_0. Then at time t later, the point P has moved to a new position, somewhere in V_t, via some path.



We want to be able to do one of two things:

1. Given initial position of P, determine current position of P,
2. Given current position of P, determine initial position of P.

These two points of view, give rise to the material and spatial description of a material.

In the material description, quantities are expressed in terms of initial position $\underset{\sim}{X}$, and we look forward to follow how the quantities are deformed. This is also known as the *Lagrangian description*.

In the spatial description, quantities are expressed in terms of the current deformed position $\underset{\sim}{x}$, and we look backwards to see how material got to its current deformation. This is also known as the *Eulerian description*.

3.1.1 Lagrangian description

This description follows the initial configuration of particles, $\underset{\sim}{X}$, as they move to the current configuration of particles, $\underset{\sim}{x}$, caused by the deforming continuum material.

In this case, quantities are expressed as functions of the material coordinates $\underset{\sim}{X}$ and time t, i.e.,

$$T = T(\underset{\sim}{X}, t), \qquad \underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{X}, t),$$

where T is the temperature and $\underset{\sim}{v}$ is the velocity.

Recall: This is also commonly known as the material description.

3.1.2 Eulerian description

This description follows the current configuration of particles, $\underset{\sim}{x}$, and traces the deformation back to the initial configuration of particles, $\underset{\sim}{X}$.

In this case, quantities are expressed as functions of the deformed coordinates $\underset{\sim}{x}$ and time t, i.e.,

$$\rho = \rho(\underset{\sim}{x}, t), \qquad \underset{\sim}{a} = \underset{\sim}{a}(\underset{\sim}{x}, t),$$

where ρ is the density and $\underset{\sim}{a}$ is the acceleration.

Recall: This is also commonly known as the spatial description.

Exercise 3.1:

Given the definitions above, state whether the following quantities are a Lagrangian or Eulerian description.

Answer:

1. $f(\underset{\sim}{X})$.
2. $f(\underset{\sim}{x})$.
3. $g(\underset{\sim}{x}, t)$.
4. $\dfrac{\partial \Phi}{\partial X^3}$.
5. $\rho(\underset{\sim}{X})$.
6. $\eta(\underset{\sim}{x}(\underset{\sim}{X}))$.

□

3.1.3 Deformation gradient

Definition 3.4:

If $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t)$, then F^i_j is the *deformation gradient tensor*, and is given by

$$F^i_j = \frac{\partial x^i}{\partial X^j}.$$

□

Thus, for a Lagrangian description, i.e., $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t)$, the differential $\mathrm{d}x^i$ is given by

$$\begin{aligned} \mathrm{d}x^i &= \frac{\partial x^i}{\partial X^1}\mathrm{d}X^1 + \frac{\partial x^i}{\partial X^2}\mathrm{d}X^2 + \frac{\partial x^i}{\partial X^3}\mathrm{d}X^3, \\ \text{i.e.,} \qquad \mathrm{d}x^i &= F^i_j \mathrm{d}X^j. \end{aligned}$$

Note:

Recall,

$$J = \left|\frac{\partial x^i}{\partial X^j}\right| \qquad \Rightarrow \qquad J = |F^i_j|,$$

where J is the usual Jacobian determinant.

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Now, consider

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t) = \underset{\sim}{x}(\underset{\sim}{X}(\underset{\sim}{x}, t), t),$$

so that

$$\begin{aligned} \frac{\partial x^i}{\partial x^j} = \delta^i_j &= \frac{\partial x^i}{\partial X^1}\frac{\partial X^1}{\partial x^j} + \frac{\partial x^i}{\partial X^2}\frac{\partial X^2}{\partial x^j} + \frac{\partial x^i}{\partial X^3}\frac{\partial X^3}{\partial x^j}, \\ &= F^i_k \frac{\partial X^k}{\partial x^j}. \end{aligned}$$

Thus,

$$\delta^i_j = F^i_k \frac{\partial X^k}{\partial x^j},$$

so that if we let

$$F^{k-1}_j = \frac{\partial X^k}{\partial x^j} \qquad \Rightarrow \qquad F^i_k \, F^{k-1}_j = \delta^i_j.$$

Note:

1. Clearly, $F^i_k \, F^{k-1}_j$ is matrix multiplication, where the matrix form of the equation is $\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$.
2. As a result:

$$\begin{aligned} \mathrm{d}x^i &= F^i_j \mathrm{d}X^j, \qquad \text{or} \qquad \mathrm{d}\underset{\sim}{x} = \mathbf{F} \cdot \mathrm{d}\underset{\sim}{X}, \qquad \text{and} \\ \mathrm{d}X^k &= F^{k-1}_j \mathrm{d}x^j, \qquad \text{or} \qquad \mathrm{d}\underset{\sim}{X} = \mathbf{F}^{-1} \cdot \mathrm{d}\underset{\sim}{x}. \end{aligned}$$

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Exercise 3.2:

Consider the usual cylindrical polar coordinates, i.e.,

$$x^1 = X^1 \cos X^2 \qquad x^2 = X^1 \sin X^2 \qquad x^3 = X^3.$$

Find F and F^{-1}.

✠

Answer:

□

Summary

In this lecture, we ...

- started Part 2: Foundation of continuum mechanics
- introduced Lagrangian and Eulerian description

Coming up

In the next lecture, we ...

- introduce the Lagrangian strain tensor
- introduce Green's deformation tensor

Homework Exercise 3.1:

1. Consider the elastic rectangular prism given in Example 3.1. If the forces F were applied in the opposite directions, then what would be the resulting strains?

2. Consider the coordinate system

   $$x^1 = X^1 \sinh X^2 \qquad x^2 = X^1 \cosh X^2 \qquad x^3 = X^3.$$

   Find F and F^{-1}.