# Review

In the previous lecture, we ...

- introduced the double dot product of two dyadics
- introduced the concept of polyads

# Aims

In this lecture, we will ...

- introduce scalar and vector invariants
- summarize tensors



**Exercise 2.32:**

If $\mathbf{D} = \underset{\sim}{i} \otimes \underset{\sim}{j} + \underset{\sim}{k} \otimes \underset{\sim}{k}$ and $\mathbf{F} = 3\underset{\sim}{i} \otimes \underset{\sim}{j} - 2\underset{\sim}{j} \otimes \underset{\sim}{i}$, then find $\mathbf{D} : \mathbf{F}$ and $\mathbf{D} \cdot\cdot\, \mathbf{F}$.

✠

**Answer:**



□

## 2.10 Scalar and vector invariants of a dyadic

Here we are going to consider how to find the scalar and vector invariants of a dyadic.

Let $\underset{\sim}{a}_i$ and $\underset{\sim}{b}_i$ denote arbitrary vectors for $i = 1, \ldots, n$, such that

$$\mathbf{P} = \underset{\sim}{a}_1 \otimes \underset{\sim}{b}_1 + \underset{\sim}{a}_2 \otimes \underset{\sim}{b}_2 + \cdots + \underset{\sim}{a}_n \otimes \underset{\sim}{b}_n,$$

is a dyadic.



Then, the *first scalar invariant of* $\mathbf{P}$ is

$$p_1 = \underset{\sim}{a}_1 \cdot \underset{\sim}{b}_1 + \underset{\sim}{a}_2 \cdot \underset{\sim}{b}_2 + \cdots + \underset{\sim}{a}_n \cdot \underset{\sim}{b}_n,$$

where the $\otimes$ operator has been replaced by the dot product operator.

Clearly, the first scalar invariant is a scalar (due to the dot product), and is invariant in regards to the coordinate system.

Further, the *first vector invariant of* $\mathbf{P}$ is

$$\underset{\sim}{p}_1 = \underset{\sim}{a}_1 \times \underset{\sim}{b}_1 + \underset{\sim}{a}_2 \times \underset{\sim}{b}_2 + \cdots + \underset{\sim}{a}_n \times \underset{\sim}{b}_n,$$

where the $\otimes$ operator has been replaced by the cross product operator.

Because the $\otimes$ operator has been replaced by the $\times$ operator, then the result is a vector, which will be invariant in regards to the coordinate system.

**Example 2.29:**

In Exercise 2.24, we found the direction cosines

$$\alpha^i_j = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

between two coordinate systems $(x^1, x^2, x^3)$ and $(X^1, X^2, X^3)$, where $x^i = \alpha^i_j X^j$. If $\underset{\sim}{a}_1(\underset{\sim}{X}) = (0, 1, -1)$ and $\underset{\sim}{b}_1(\underset{\sim}{X}) = (1, \sqrt{2}, 0)$, then show the first scalar and vector invariants of $\underset{\sim}{a}_1 \otimes \underset{\sim}{b}_1$ are indeed invariant with respect to the two coordinate systems. ✠



**Answer:**

If

$$\mathbf{P}(\underset{\sim}{X}) = \underset{\sim}{a}_1 \otimes \underset{\sim}{b}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \sqrt{2} & 0 \\ -1 & -\sqrt{2} & 0 \end{pmatrix},$$

then

$$p_1(\underset{\sim}{X}) = \underset{\sim}{a}_1 \cdot \underset{\sim}{b}_1 = (0, 1, -1) \cdot (1, \sqrt{2}, 0) = \sqrt{2},$$

while

$$\underset{\sim}{p}_1(\underset{\sim}{X}) = \underset{\sim}{a}_1 \times \underset{\sim}{b}_1 = \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ 0 & 1 & -1 \\ 1 & \sqrt{2} & 0 \end{vmatrix} = (\sqrt{2}, -1, -1).$$

Thus, we know $p_1$ and $\underset{\sim}{p}_1$ in terms of $\underset{\sim}{X}$.



To check to see if they are invariant, we first need to convert $\underset{\sim}{a}_1$ and $\underset{\sim}{b}_1$ into $\underset{\sim}{x}$. Thus,

$$\bar{\underset{\sim}{a}}_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix},$$

$$\bar{\underset{\sim}{b}}_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 0 \end{pmatrix}.$$

Hence,

$$\mathbf{P}(\underset{\sim}{x}) = \bar{\underset{\sim}{a}}_1 \otimes \bar{\underset{\sim}{b}}_1 = \begin{pmatrix} \sqrt{2} & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$



Thus,

$$p_1(\underset{\sim}{x}) = \bar{\underset{\sim}{a}}_1 \cdot \bar{\underset{\sim}{b}}_1 = (\sqrt{2}, 0, 0) \cdot (1, \sqrt{2}, 0) = \sqrt{2},$$

while

$$\underset{\sim}{p}_1(\underset{\sim}{x}) = \bar{\underset{\sim}{a}}_1 \times \bar{\underset{\sim}{b}}_1 = \begin{vmatrix} \bar{\underset{\sim}{e}}_1 & \bar{\underset{\sim}{e}}_2 & \bar{\underset{\sim}{e}}_3 \\ \sqrt{2} & 0 & 0 \\ 1 & \sqrt{2} & 0 \end{vmatrix} = (0, 0, 2).$$

Hence, while we have shown that

$$p_1(\underset{\sim}{X}) = p_1(\underset{\sim}{x}) = \sqrt{2},$$

at first glance it seems that

$$(\sqrt{2}, -1, -1) = \underset{\sim}{p}_1(\underset{\sim}{X}) \neq \underset{\sim}{p}_1(\underset{\sim}{x}) = (0, 0, 2).$$

However, $\underset{\sim}{p}_1(\underset{\sim}{X})$ corresponds to the basis vectors $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$, while $\underset{\sim}{p}_1(\underset{\sim}{x})$ corresponds to the basis vectors $(\bar{\underset{\sim}{e}}_1, \bar{\underset{\sim}{e}}_2, \bar{\underset{\sim}{e}}_3)$.

Thus, we know that $\bar{\underset{\sim}{e}}_i = \alpha_i^j \underset{\sim}{e}_j$, so that

$$\begin{aligned}
\underset{\sim}{p}_1(\underset{\sim}{x}) &= (0, 0, 2), \\
&= 2\bar{\underset{\sim}{e}}_3, \\
&= 2\left(\frac{1}{\sqrt{2}}\underset{\sim}{e}_1 - \frac{1}{2}\underset{\sim}{e}_2 - \frac{1}{2}\underset{\sim}{e}_3\right), \\
&= \sqrt{2}\underset{\sim}{e}_1 - \underset{\sim}{e}_2 - \underset{\sim}{e}_3, \\
&= (\sqrt{2}, -1, -1), \\
&= \underset{\sim}{p}_1(\underset{\sim}{X}).
\end{aligned}$$

□



**Exercise 2.33:**

Let

$$\mathbf{P} = \underset{\sim}{i} \otimes \underset{\sim}{j} + \underset{\sim}{j} \otimes \underset{\sim}{k} + \underset{\sim}{k} \otimes \underset{\sim}{k}.$$

Find the first scalar and vector invariant.

✠

**Answer:**



□

**Note:**

It is possible to find *higher order* scalar and vector invariants. However, we will not consider them here in MATH312.

♪



## 2.11 Summary of Tensors

In this chapter, we have introduced *tensors* and related concepts.

In this lecture, I will *briefly* summarize what we have learnt about tensors.

- The Einstein Summation convention
  - if a repeated index occurs exactly twice in the same term, then sum over the index is implied
  - $a_i x_i = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$
- If the range of a free (or dummy) index is not specified, then implicity assume a range of $1, \ldots, 3$
  - $a_i x_i = a_1 x_1 + a_2 x_2 + a_3 x_3$

- The Kronecker delta symbol and the permutation symbol
  - $\delta_{ij}$ and $\varepsilon_{ijk}$
- Base vectors
  - $\underset{\sim}{e}_i$
  - properties: orthogonal? unit vectors? other?
  - tangential and gradient reciprocal basis vectors
- Representing a vector using indicial notation
  - $\underset{\sim}{v} = v^i \underset{\sim}{e}_i$
  - contravariant and covariant components
  - meaning of components
  - physical components



- Operations on vectors
  - cross product
  - dot product
  - scalar triple product
  - the del operator, $\underset{\sim}{\nabla}$
  - derivative shorthand notation: $x_{i,j}$
- Dyads and dyadics
  - intermediate vector product
  - cross product
  - dot product
  - matrix multiplication
  - select elements using $\underset{\sim}{e}_i$'s



- Direction cosines
  - relationship between axes
  - relationship between base vectors
  - orthogonality condition
  - rotation of axes
  - identity: $\alpha^T \alpha = \underset{\sim}{I}$
- Classification of tensors
  - order
  - type
  - transformation laws
  - relative and absolute
  - symmetric and skew-symmetric



- Basic tensor operations
  - equality
  - addition/subtration
  - multiplication (outer product)
  - contraction
- Curvilinear coordinates
  - $x^k = x^k(X^1, X^2, X^3)$
  - $X^k = X^k(x^1, x^2, x^3)$
  - Jacobian determinant: $J \neq 0 \Rightarrow$ inverse exists
  - Cartesian coordinates: $(z^1, z^2, z^3) = (x, y, z)$

- Calculation of derivatives
  - $\Phi(X^1, X^2, X^3) \Rightarrow \dfrac{\partial \Phi}{\partial x^i} = \dfrac{\partial \Phi}{\partial X^j} \dfrac{\partial X^j}{\partial x^i}$
  - importance?
- The metric tensor $g_{ij}$
  - definition
  - properties
  - $g = J^2$
  - relationship with direction cosines
  - relationship with tangential basis vectors
  - relationship with a position vector



- The conjugate metric tensor $g^{jk}$
  - definition
  - properties
  - relationship with $g_{ij}$
  - diagonal?
- Rasing and lower indices
  - lower index using $g_{ij}$
  - raise index using $g^{jk}$
- The physical components
  - in terms of $g_{ij}$ and $g^{jk}$
  - second-order tensors



- Other operations on dyads
  - conjugate dyadic $A_c$
  - idemfactor $\mathbf{J}$
  - double dot product :
  - other double dot and cross products
- Polyads
  - triads
- Scalar and vector invariants of dyads.



## Summary

In this lecture, we ...

- introduce scalar and vector invariants
- summarize tensors

## Coming up

In the next lecture, we ...

- start Part 2: Foundation of continuum mechanics
- introduce Lagrangian and Eulerian description

**Homework Exercise 2.13:**

1. Let $\underset{\sim}{f} = f_1\underset{\sim}{i} + f_2\underset{\sim}{j} + f_3\underset{\sim}{k}$, and consider the dyad

$$\underset{\sim}{\nabla} \otimes \underset{\sim}{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} & \frac{\partial f_3}{\partial y} \\ \frac{\partial f_1}{\partial z} & \frac{\partial f_2}{\partial z} & \frac{\partial f_3}{\partial z} \end{pmatrix}.$$

   Find the first scalar and vector invariants.

2. Write your own summary of Part 1: Tensors, including formulaes, definitions and examples.