Review

In the previous lecture, we ...

- considered an example on direction cosines
- considered more operations on dyads

Aims

In this lecture, we will ...

- introduce the double dot product of two dyadics
- introduce the concept of polyads

Exercise 2.28:
Let $\underset{\sim}{u} = (1,0,1)$, $\underset{\sim}{v} = (0,1,1)$ and $\underset{\sim}{w} = (1,1,1)$. Find $\mathbf{A}$ and $\mathbf{B}$ if

$$\mathbf{A} = \underset{\sim}{u} \otimes \underset{\sim}{v}, \qquad \mathbf{B} = \underset{\sim}{w} \otimes \underset{\sim}{w}.$$

Further, find $\mathbf{A} \cdot \mathbf{B}$.

✠

Answer:

□

2.8.8 The conjugate dyadic and the idemfactor

We have shown previously that a dyadic $\mathbf{A}$ can be represented by

$$\begin{aligned}\mathbf{A} = A_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = {} & A_{11}\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + A_{12}\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2 + A_{13}\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_3 \\ & + A_{21}\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_1 + A_{22}\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2 + A_{23}\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_3 \\ & + A_{31}\underset{\sim}{e}_3 \otimes \underset{\sim}{e}_1 + A_{32}\underset{\sim}{e}_3 \otimes \underset{\sim}{e}_2 + A_{33}\underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3,\end{aligned}$$

where the *conjugate dyadic* $\mathbf{A}_c$ is defined by a transposition of the unit vectors in $\mathbf{A}$, i.e.,

$$\begin{aligned}\mathbf{A}_c = A_{ij}\underset{\sim}{e}_j \otimes \underset{\sim}{e}_i = {} & A_{11}\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + A_{12}\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_1 + A_{13}\underset{\sim}{e}_3 \otimes \underset{\sim}{e}_1 \\ & + A_{21}\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2 + A_{22}\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2 + A_{23}\underset{\sim}{e}_3 \otimes \underset{\sim}{e}_2 \\ & + A_{31}\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_3 + A_{32}\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_3 + A_{33}\underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3.\end{aligned}$$

If a dyadic equals its conjugate, i.e., $\mathbf{A} = \mathbf{A}_c$, then $A_{ij} = A_{ji}$, and the dyadic is symmetric.

Similarly, if a dyadic equals the negative of its conjugate, i.e., $\mathbf{A} = -\mathbf{A}_c$, then $A_{ij} = -A_{ji}$, and the dyadic is skew-symmetric.

A special dyadic, called the *identical dyadic* or *idemfactor*, is defined by

$$\mathbf{J} = \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2 + \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3 = \underset{\sim}{e}_i \otimes \underset{\sim}{e}_i.$$

This dyadic has the property that pre or post dot product multiplication of $\mathbf{J}$ with a vector $\underset{\sim}{v}$ returns the same vector $\underset{\sim}{v}$.

For example, if $\underset{\sim}{v} = v^1\underset{\sim}{e}_1 + v^2\underset{\sim}{e}_2 + v^3\underset{\sim}{e}_3$, then

$$\underset{\sim}{v} \cdot \mathbf{J} = (v^1\underset{\sim}{e}_1 + v^2\underset{\sim}{e}_2 + v^3\underset{\sim}{e}_3) \cdot (\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2 + \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3),$$

$$\begin{aligned}
&= v^1\underset{\sim}{e}_1 \cdot (\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2 + \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3) \\
&+ v^2\underset{\sim}{e}_2 \cdot (\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2 + \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3) \\
&+ v^3\underset{\sim}{e}_3 \cdot (\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2 + \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3),
\end{aligned}$$

$$\begin{aligned}
&= v^1[(\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_1)\underset{\sim}{e}_1 + (\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_2)\underset{\sim}{e}_2 + (\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_3)\underset{\sim}{e}_3] \\
&+ v^2[(\underset{\sim}{e}_2 \cdot \underset{\sim}{e}_1)\underset{\sim}{e}_1 + (\underset{\sim}{e}_2 \cdot \underset{\sim}{e}_2)\underset{\sim}{e}_2 + (\underset{\sim}{e}_2 \cdot \underset{\sim}{e}_3)\underset{\sim}{e}_3] \\
&+ v^3[(\underset{\sim}{e}_3 \cdot \underset{\sim}{e}_1)\underset{\sim}{e}_1 + (\underset{\sim}{e}_3 \cdot \underset{\sim}{e}_2)\underset{\sim}{e}_2 + (\underset{\sim}{e}_3 \cdot \underset{\sim}{e}_3)\underset{\sim}{e}_3],
\end{aligned}$$

$$= v^1\underset{\sim}{e}_1 + v^2\underset{\sim}{e}_2 + v^3\underset{\sim}{e}_3 = \underset{\sim}{v}.$$

And similarly for $\mathbf{J} \cdot \underset{\sim}{v}$.

Exercise 2.29:
Let $\mathbf{A}$ and $\mathbf{B}$ be two dyadics such that

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & -2 \\ -1 & 1 & 0 \end{pmatrix}.$$

Find $\mathbf{A}_c$, $\mathbf{B}_c$ and $\mathbf{C} = \mathbf{A} + \mathbf{B}_c$. Check to see if any of these dyadics are symmetric or skew-symmetric.

✠

Answer:

□

2.8.9 The double dot product

A dyadic operation often used in physics and chemistry is the double dot product, $\mathbf{A} : \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are both dyadics. Both dyadics are expanded using the distributive law of multiplication, and then each unit dyad pair $\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j : \underset{\sim}{e}_m \otimes \underset{\sim}{e}_n$ are combined according to the rule

$$\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j : \underset{\sim}{e}_m \otimes \underset{\sim}{e}_n = (\underset{\sim}{e}_i \cdot \underset{\sim}{e}_m)(\underset{\sim}{e}_j \cdot \underset{\sim}{e}_n).$$

Example 2.27:

If $\mathbf{A} = A_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$ and $\mathbf{B} = B_{mn}\underset{\sim}{e}_m \otimes \underset{\sim}{e}_n$, then the double dot product $\mathbf{A} : \mathbf{B}$ is calculated as follows.

$$\begin{aligned}
\mathbf{A} : \mathbf{B} &= (A_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) : (B_{mn}\underset{\sim}{e}_m \otimes \underset{\sim}{e}_n), \\
&= A_{ij}B_{mn}(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j : \underset{\sim}{e}_m \otimes \underset{\sim}{e}_n), \\
&= A_{ij}B_{mn}(\underset{\sim}{e}_i \cdot \underset{\sim}{e}_m)(\underset{\sim}{e}_j \cdot \underset{\sim}{e}_n), \\
&= A_{ij}B_{mn}\delta_{im}\delta_{jn}, \\
&= A_{mj}B_{mj}, \\
&= A_{11}B_{11} + A_{12}B_{12} + A_{13}B_{13} \\
&+ A_{21}B_{21} + A_{22}B_{22} + A_{23}B_{23} \\
&+ A_{31}B_{31} + A_{32}B_{32} + A_{33}B_{33}.
\end{aligned}$$

□

Exercise 2.30:
Let $\mathbf{A} = A_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$ denote a dyadic. Find $\mathbf{A} : \mathbf{A}_c$.

✠

Answer:

□

Note:

1. When considering dyads, the order of the vectors is important, as the intermediate vector product of two vectors is non-commutative, i.e.,
$$\underset{\sim}{u} \otimes \underset{\sim}{v} \neq \underset{\sim}{v} \otimes \underset{\sim}{u},$$
in general.
2. We know that the dot product of two vectors gives a scalar, i.e., if $\underset{\sim}{u} = u^i\underset{\sim}{e}_i$ and $\underset{\sim}{v} = v^j\underset{\sim}{e}_j$, then
$$\underset{\sim}{v} \cdot \underset{\sim}{u} = v^1u^1 + v^2u^2 + v^3u^3 = \text{scalar}.$$
Similarly, the double dot product of two dyadics gives a scalar, i.e., if $\mathbf{A} = A_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$ and $\mathbf{B} = B_{mn}\underset{\sim}{e}_m \otimes \underset{\sim}{e}_n$, then
$$\mathbf{A} : \mathbf{B} = A_{mj}B_{mj} = \text{scalar}.$$

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2.8.10 Other double dot and cross products

We have just seen the double dot product of two dyadics. Other double dot and cross products do exists, namely if $\mathbf{A} = \underset{\sim}{a} \otimes \underset{\sim}{b}$ and $\mathbf{C} = \underset{\sim}{c} \otimes \underset{\sim}{d}$, then

$$\mathbf{A} : \mathbf{B} = (\underset{\sim}{a} \otimes \underset{\sim}{b}) : (\underset{\sim}{c} \otimes \underset{\sim}{d}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})(\underset{\sim}{b} \cdot \underset{\sim}{d}) = \lambda, \qquad \text{a scalar,}$$

$$\mathbf{A} \overset{\times}{\cdot} \mathbf{B} = (\underset{\sim}{a} \otimes \underset{\sim}{b}) \overset{\times}{\cdot} (\underset{\sim}{c} \otimes \underset{\sim}{d}) = (\underset{\sim}{a} \times \underset{\sim}{c})(\underset{\sim}{b} \cdot \underset{\sim}{d}) = \underset{\sim}{u}, \qquad \text{a vector,}$$

$$\mathbf{A} \underset{\times}{\cdot} \mathbf{B} = (\underset{\sim}{a} \otimes \underset{\sim}{b}) \underset{\times}{\cdot} (\underset{\sim}{c} \otimes \underset{\sim}{d}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})(\underset{\sim}{b} \times \underset{\sim}{d}) = \underset{\sim}{v}, \qquad \text{a vector,}$$

$$\mathbf{A} \underset{\times}{\times} \mathbf{B} = (\underset{\sim}{a} \otimes \underset{\sim}{b}) \underset{\times}{\times} (\underset{\sim}{c} \otimes \underset{\sim}{d}) = (\underset{\sim}{a} \times \underset{\sim}{c}) \otimes (\underset{\sim}{b} \times \underset{\sim}{d}) = \underset{\sim}{u} \otimes \underset{\sim}{v}, \quad \text{a dyadic.}$$

Note:

It is possible to have alternate definitions for the above double dot and cross products. For example,

$$\mathbf{A} \cdot\cdot\, \mathbf{B} = (\underset{\sim}{a} \otimes \underset{\sim}{b}) \cdot\cdot\, (\underset{\sim}{c} \otimes \underset{\sim}{d}) = (\underset{\sim}{a} \cdot \underset{\sim}{d})(\underset{\sim}{b} \cdot \underset{\sim}{c}) = \gamma, \qquad \text{a scalar.}$$

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Example 2.28:

Let

$$\mathbf{D} = 3\underset{\sim}{i} \otimes \underset{\sim}{i} + 2\underset{\sim}{j} \otimes \underset{\sim}{j} - \underset{\sim}{j} \otimes \underset{\sim}{k} + 5\underset{\sim}{k} \otimes \underset{\sim}{k},$$
$$\mathbf{F} = 4\underset{\sim}{i} \otimes \underset{\sim}{k} + 6\underset{\sim}{j} \otimes \underset{\sim}{j} - 3\underset{\sim}{k} \otimes \underset{\sim}{j} + \underset{\sim}{k} \otimes \underset{\sim}{k}.$$

Compute and compare $\mathbf{D} : \mathbf{F}$ and $\mathbf{D} \cdot\cdot\, \mathbf{F}$.

✠

Answer:

To find $\mathbf{D} : \mathbf{F}$, consider

$$\begin{aligned}
&\mathbf{D} : \mathbf{F} \\
&= (3\underset{\sim}{i} \otimes \underset{\sim}{i} + 2\underset{\sim}{j} \otimes \underset{\sim}{j} - \underset{\sim}{j} \otimes \underset{\sim}{k} + 5\underset{\sim}{k} \otimes \underset{\sim}{k}) : (4\underset{\sim}{i} \otimes \underset{\sim}{k} + 6\underset{\sim}{j} \otimes \underset{\sim}{j} - 3\underset{\sim}{k} \otimes \underset{\sim}{j} + \underset{\sim}{k} \otimes \underset{\sim}{k}), \\
&= 3\underset{\sim}{i} \otimes \underset{\sim}{i} : (4\underset{\sim}{i} \otimes \underset{\sim}{k} + 6\underset{\sim}{j} \otimes \underset{\sim}{j} - 3\underset{\sim}{k} \otimes \underset{\sim}{j} + \underset{\sim}{k} \otimes \underset{\sim}{k}) \\
&\quad + 2\underset{\sim}{j} \otimes \underset{\sim}{j} : (4\underset{\sim}{i} \otimes \underset{\sim}{k} + 6\underset{\sim}{j} \otimes \underset{\sim}{j} - 3\underset{\sim}{k} \otimes \underset{\sim}{j} + \underset{\sim}{k} \otimes \underset{\sim}{k}) \\
&\quad - \underset{\sim}{j} \otimes \underset{\sim}{k} : (4\underset{\sim}{i} \otimes \underset{\sim}{k} + 6\underset{\sim}{j} \otimes \underset{\sim}{j} - 3\underset{\sim}{k} \otimes \underset{\sim}{j} + \underset{\sim}{k} \otimes \underset{\sim}{k}) \\
&\quad + 5\underset{\sim}{k} \otimes \underset{\sim}{k} : (4\underset{\sim}{i} \otimes \underset{\sim}{k} + 6\underset{\sim}{j} \otimes \underset{\sim}{j} - 3\underset{\sim}{k} \otimes \underset{\sim}{j} + \underset{\sim}{k} \otimes \underset{\sim}{k}), \\
&= \ 0 \\
&\quad + (2\underset{\sim}{j} \otimes \underset{\sim}{j}) : (6\underset{\sim}{j} \otimes \underset{\sim}{j}) \\
&\quad - 0 \\
&\quad + (5\underset{\sim}{k} \otimes \underset{\sim}{k}) : (\underset{\sim}{k} \otimes \underset{\sim}{k}),
\end{aligned}$$

Thus,

$$\mathbf{D} : \mathbf{F} = (2 \cdot 6)(1 \cdot 1) + (5 \cdot 1)(1 \cdot 1) = 17$$

Next, $\mathbf{D} \cdot\cdot\, \mathbf{F}$ is given by

$$\begin{aligned}
&\mathbf{D} \cdot\cdot\, \mathbf{F} \\
&= (3\underset{\sim}{i} \otimes \underset{\sim}{i} + 2\underset{\sim}{j} \otimes \underset{\sim}{j} - \underset{\sim}{j} \otimes \underset{\sim}{k} + 5\underset{\sim}{k} \otimes \underset{\sim}{k}) \cdot\cdot\, (4\underset{\sim}{i} \otimes \underset{\sim}{k} + 6\underset{\sim}{j} \otimes \underset{\sim}{j} - 3\underset{\sim}{k} \otimes \underset{\sim}{j} + \underset{\sim}{k} \otimes \underset{\sim}{k}), \\
&= 3\underset{\sim}{i} \otimes \underset{\sim}{i} \cdot\cdot\, (4\underset{\sim}{i} \otimes \underset{\sim}{k} + 6\underset{\sim}{j} \otimes \underset{\sim}{j} - 3\underset{\sim}{k} \otimes \underset{\sim}{j} + \underset{\sim}{k} \otimes \underset{\sim}{k}) \\
&\quad + 2\underset{\sim}{j} \otimes \underset{\sim}{j} \cdot\cdot\, (4\underset{\sim}{i} \otimes \underset{\sim}{k} + 6\underset{\sim}{j} \otimes \underset{\sim}{j} - 3\underset{\sim}{k} \otimes \underset{\sim}{j} + \underset{\sim}{k} \otimes \underset{\sim}{k}) \\
&\quad - \underset{\sim}{j} \otimes \underset{\sim}{k} \cdot\cdot\, (4\underset{\sim}{i} \otimes \underset{\sim}{k} + 6\underset{\sim}{j} \otimes \underset{\sim}{j} - 3\underset{\sim}{k} \otimes \underset{\sim}{j} + \underset{\sim}{k} \otimes \underset{\sim}{k}) \\
&\quad + 5\underset{\sim}{k} \otimes \underset{\sim}{k} \cdot\cdot\, (4\underset{\sim}{i} \otimes \underset{\sim}{k} + 6\underset{\sim}{j} \otimes \underset{\sim}{j} - 3\underset{\sim}{k} \otimes \underset{\sim}{j} + \underset{\sim}{k} \otimes \underset{\sim}{k}), \\
&= 0 \\
&\quad + (2\underset{\sim}{j} \otimes \underset{\sim}{j}) \cdot\cdot\, (6\underset{\sim}{j} \otimes \underset{\sim}{j}) \\
&\quad + (-\underset{\sim}{j} \otimes \underset{\sim}{k}) \cdot\cdot\, (-3\underset{\sim}{k} \otimes \underset{\sim}{j}) \\
&\quad + (5\underset{\sim}{k} \otimes \underset{\sim}{k}) \cdot\cdot\, (\underset{\sim}{k} \otimes \underset{\sim}{k}),
\end{aligned}$$

Thus,

$$\mathbf{D} \cdot\cdot\, \mathbf{F} = (2\cdot 1)(1\cdot 6) + (-1\cdot 1)(1\cdot -3) + (5\cdot 1)(1\cdot 1) = 20.$$

Therefore, in comparison, we find

$$\mathbf{D} : \mathbf{F} = 17,$$
$$\mathbf{D} \cdot\cdot\, \mathbf{F} = 20,$$

and hence different definitions used for the double dot product can lead to different values.

□

Exercise 2.31:

If $\mathbf{D} = 2\underset{\sim}{i} \otimes \underset{\sim}{j} - \underset{\sim}{k} \otimes \underset{\sim}{k}$ and $\mathbf{F} = 3\underset{\sim}{i} \otimes \underset{\sim}{j} - \underset{\sim}{j} \otimes \underset{\sim}{i} + 6\underset{\sim}{k} \otimes \underset{\sim}{k}$, then find $\mathbf{D} : \mathbf{F}$ and $\mathbf{D} \cdot\cdot\, \mathbf{F}$.

✠

Answer:

□

2.9 Polyads

Dyads are a special case of *polyads*. Another common special case are *triads*, which are of the form

$$\mathbf{A} = A_{ijk}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k.$$

In general, a polyad is a mixed tensor that has the representation

$$\mathbf{T} = T^{ij\ldots k}_{lm\ldots n}\; \underset{\sim}{E}_i \underset{\sim}{E}_j \ldots \underset{\sim}{E}_k\; \underset{\sim}{E}^l \underset{\sim}{E}^m \ldots \underset{\sim}{E}^n,$$

where $\underset{\sim}{E}_i$ and $\underset{\sim}{E}_m$ denote the tangential and gradient basis vectors, respectively.

We have been looking at dyads in reference to the Cartesian coordinates, which means that

$$\underset{\sim}{E}_i = \underset{\sim}{E}^i = \underset{\sim}{e}_i.$$

For MATH312, you will only be asked about dyads in particular, and polyads in general, in reference to the Cartesian coordinates.

Summary

In this lecture, we ...

- introduced the double dot product of two dyadics
- introduced the concept of polyads

Coming up

In the next lecture, we will ...

- introduce scalar and vector invariants
- summarize tensors

Homework Exercise 2.12:

1. Let $\mathbf{A}$ and $\mathbf{B}$ be two dyadics such that

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -1 & 2 & 1 \\ -1 & -2 & 0 \\ -3 & 3 & 0 \end{pmatrix}.$$

   Find $\mathbf{A}_c$, $\mathbf{B}_c$ and $\mathbf{C} = \mathbf{A} + \mathbf{B}_c$. Check to see if any of these dyadics are symmetric or skew-symmetric.

2. If $\mathbf{D} = 3\underset{\sim}{i} \otimes \underset{\sim}{i} - \underset{\sim}{i} \otimes \underset{\sim}{k} + 2\underset{\sim}{j} \otimes \underset{\sim}{k}$ and $\mathbf{F} = -\underset{\sim}{i} \otimes \underset{\sim}{i} + 3\underset{\sim}{j} \otimes \underset{\sim}{k} + 2\underset{\sim}{k} \otimes \underset{\sim}{i}$, then find $\mathbf{D} : \mathbf{F}$, $\mathbf{F} : \mathbf{D}$, $\mathbf{F} \cdot\cdot\, \mathbf{D}$, $\mathbf{D}_c : \mathbf{F}$ and $\mathbf{D} \overset{\times}{\cdot} \mathbf{F}_c$.