Review

In the previous lecture, we ...

- introduced direction cosines
- introduced dyads

Aims

In this lecture, we will ...

- consider an example on direction cosines
- consider more operations on dyads

Example 2.25:

Calculate the direction cosines $\bar{\alpha}^i_j$ for the pair of Cartesian coordinate systems related by

$$\bar{\underset{\sim}{e}}_1 = \underset{\sim}{e}_2, \qquad \bar{\underset{\sim}{e}}_2 = \underset{\sim}{e}_3, \qquad \bar{\underset{\sim}{e}}_3 = \underset{\sim}{e}_1,$$

and sketch the the geometrical relationship between the coordinate systems. ✠

Answer:

Using direction cosines, the transformation law between the unit vectors is $\bar{\underset{\sim}{e}}_i = \bar{\alpha}^j_i \underset{\sim}{e}_j$, so that

$$\begin{aligned}
\bar{\underset{\sim}{e}}_1 &= \bar{\alpha}^1_1 \underset{\sim}{e}_1 + \bar{\alpha}^2_1 \underset{\sim}{e}_2 + \bar{\alpha}^3_1 \underset{\sim}{e}_3 = \underset{\sim}{e}_2,\\
\bar{\underset{\sim}{e}}_2 &= \bar{\alpha}^1_2 \underset{\sim}{e}_1 + \bar{\alpha}^2_2 \underset{\sim}{e}_2 + \bar{\alpha}^3_2 \underset{\sim}{e}_3 = \underset{\sim}{e}_3,\\
\bar{\underset{\sim}{e}}_3 &= \bar{\alpha}^1_3 \underset{\sim}{e}_1 + \bar{\alpha}^2_3 \underset{\sim}{e}_2 + \bar{\alpha}^3_3 \underset{\sim}{e}_3 = \underset{\sim}{e}_1.
\end{aligned}$$

Thus,

$$\bar{\alpha}^2_1 = \bar{\alpha}^3_2 = \bar{\alpha}^1_3 = 1,$$

where all the other $\bar{\alpha}^i_j$'s are zero. Therefore, the rotation matrix is

$$\bar{\alpha} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

□

Note:

1. $\bar{\alpha}^T \bar{\alpha} = \underset{\sim}{I}$
2. If $\underset{\sim}{v}$ is a vector in relation to the unbarred coordinate system, then the vector is transformed by the relationship $\bar{\underset{\sim}{v}}_i = \bar{\alpha}^j_i \underset{\sim}{v}_i$,

i.e.,

$$\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_2 \\ v_3 \\ v_1 \end{pmatrix}.$$

Thus,

$$\bar{v}_1 = v_2, \qquad \bar{v}_2 = v_3, \qquad \bar{v}_3 = v_1.$$

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2.8.3 Some properties of dyads

We have seen how to perform a dot product and a cross product on a dyadic with a dyad. These operations are based on the following dyad properties:

1. $(\underset{\sim}{a} \otimes \underset{\sim}{b}) \cdot \underset{\sim}{c} = \underset{\sim}{a}(\underset{\sim}{b} \cdot \underset{\sim}{c})$,
2. $\underset{\sim}{a} \cdot (\underset{\sim}{b} \otimes \underset{\sim}{c}) = (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$,
3. $(\underset{\sim}{a} \otimes \underset{\sim}{b}) \times \underset{\sim}{c} = \underset{\sim}{a} \otimes (\underset{\sim}{b} \times \underset{\sim}{c})$,
4. $\underset{\sim}{a} \times (\underset{\sim}{b} \otimes \underset{\sim}{c}) = (\underset{\sim}{a} \times \underset{\sim}{b}) \otimes \underset{\sim}{c}$,
5. $(\underset{\sim}{a} \otimes \underset{\sim}{b}) \cdot (c_1\underset{\sim}{u} + c_2\underset{\sim}{v}) = c_1(\underset{\sim}{a} \otimes \underset{\sim}{b}) \cdot \underset{\sim}{u} + c_2(\underset{\sim}{a} \otimes \underset{\sim}{b}) \cdot \underset{\sim}{v}$,
6. $(\underset{\sim}{a} \otimes \underset{\sim}{b}) \times (c_1\underset{\sim}{u} + c_2\underset{\sim}{v}) = c_1(\underset{\sim}{a} \otimes \underset{\sim}{b}) \times \underset{\sim}{u} + c_2(\underset{\sim}{a} \otimes \underset{\sim}{b}) \times \underset{\sim}{v}$.

Now, let $\mathbf{D}$ be a dyadic, such that

$$\mathbf{D} = \underset{\sim}{u} \otimes \underset{\sim}{v},$$

for some vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$. To select the i,jth element of $\mathbf{D}$, denoted by D_{ij}, we perform the following operations:

$$\begin{aligned} D_{ij} &= \underset{\sim}{e}_i \cdot \mathbf{D} \cdot \underset{\sim}{e}_j, \\ &= \underset{\sim}{e}_i \cdot (\underset{\sim}{u} \otimes \underset{\sim}{v}) \cdot \underset{\sim}{e}_j, \\ &= (\underset{\sim}{e}_i \cdot \underset{\sim}{u})(\underset{\sim}{v} \cdot \underset{\sim}{e}_j), \end{aligned}$$

or, in other words, the i,jth element of $\mathbf{D}$ is found by multiplying the ith element of $\underset{\sim}{u}$ by the jth element of $\underset{\sim}{v}$.

Exercise 2.26:

Let $\underset{\sim}{u} = (1,0,2)$ and $\underset{\sim}{v} = (-1,1,0)$, so that if $\mathbf{D} = \underset{\sim}{u} \otimes \underset{\sim}{v}$, then

$$\mathbf{D} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ -2 & 2 & 0 \end{pmatrix}.$$

By taking the appropriate dot products of $\underset{\sim}{u}$ and $\underset{\sim}{v}$ with unit vectors (i.e., $D_{ij} = (\underset{\sim}{e}_i \cdot \underset{\sim}{u})(\underset{\sim}{v} \cdot \underset{\sim}{e}_j)$), find the elements D_{12} and D_{31}.

✠

Answer:

□

2.8.4 Dyad of base vectors

We can use base vectors to create dyads, i.e., $\mathbf{D} = \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$, for some i and j. In this case, if we wanted to select the m,nth element, we find

$$\begin{aligned} (\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)_{mn} &= \underset{\sim}{e}_m \cdot (\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) \cdot \underset{\sim}{e}_n, \\ &= (\underset{\sim}{e}_m \cdot \underset{\sim}{e}_i)(\underset{\sim}{e}_j \cdot \underset{\sim}{e}_n), \\ &= \delta_{mi}\delta_{jn}. \end{aligned}$$

Thus, for example, if $D_{mn} = (\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2)_{mn} = \delta_{m1}\delta_{2n}$, then

$$\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Further, this means that if T_{ij} denotes the nine elements of $\mathbf{T}$, then

$$\begin{aligned}\mathbf{T} &= T_{11}\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + T_{12}\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2 + T_{13}\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_3 + \cdots + T_{33}\underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3, \\ &= T_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j.\end{aligned}$$

Note:

1. This highlights the fact that *any* dyadic can be expressed as a *finite* sum of dyads.
2. The indices don't "balance" because we are dealing with Cartesian basis vectors, i.e., $\underset{\sim}{e}_i = \underset{\sim}{E}_i = \underset{\sim}{E}^i$.

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Example 2.26:

A simple example of dyadic representation occurs when expressing the identity matrix $\mathbf{I}$ as

$$\mathbf{I} = \delta_{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

Representing a dyadic in terms of base vectors enables us to use what we already know about the dot product and cross product of base vectors to find simple indicial expressions for the dot product and cross product of dyadics.

2.8.5 Dot product of two dyadics

Let $\mathbf{A} = A_{ij}\,\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$ and $\mathbf{B} = B_{mn}\,\underset{\sim}{e}_m \otimes \underset{\sim}{e}_n$ be two dyadics. Then

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (A_{ij}\,\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)\cdot(B_{mn}\,\underset{\sim}{e}_m \otimes \underset{\sim}{e}_n), \\ &= A_{ij}B_{mn}(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)\cdot(\underset{\sim}{e}_m \otimes \underset{\sim}{e}_n), \\ &= A_{ij}B_{mn}[(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)\cdot\underset{\sim}{e}_m] \otimes \underset{\sim}{e}_n, \\ &= A_{ij}B_{mn}\,[\underset{\sim}{e}_i(\underset{\sim}{e}_j\cdot\underset{\sim}{e}_m)] \otimes \underset{\sim}{e}_n, \\ &= A_{ij}B_{mn}\,\delta_{jm}\,\underset{\sim}{e}_i \otimes \underset{\sim}{e}_n, \\ &= A_{ij}B_{jn}\,\underset{\sim}{e}_i \otimes \underset{\sim}{e}_n, \\ \Rightarrow \qquad (\mathbf{A}\cdot\mathbf{B})_{in} &= A_{ij}B_{jn}.\end{aligned}$$

This is just matrix multiplication, where the result is a second order tensor, i.e., a dyadic.

2.8.6 Cross product of two dyadics

Let $\mathbf{A} = A_{ij}\,\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$ and $\mathbf{B} = B_{mn}\,\underset{\sim}{e}_m \otimes \underset{\sim}{e}_n$ be two dyadics. Then

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (A_{ij}\,\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)\times(B_{mn}\,\underset{\sim}{e}_m \otimes \underset{\sim}{e}_n), \\ &= A_{ij}B_{mn}\,(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)\times(\underset{\sim}{e}_m \otimes \underset{\sim}{e}_n), \\ &= A_{ij}B_{mn}\,[(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)\times\underset{\sim}{e}_m] \otimes \underset{\sim}{e}_n, \\ &= A_{ij}B_{mn}\,[\underset{\sim}{e}_i \otimes (\underset{\sim}{e}_j\times\underset{\sim}{e}_m)] \otimes \underset{\sim}{e}_n, \\ &= A_{ij}B_{mn}\,[\underset{\sim}{e}_i \otimes (\varepsilon_{jml}\,\underset{\sim}{e}_l)] \otimes \underset{\sim}{e}_n, \\ &= \varepsilon_{jml}\,A_{ij}B_{mn}\,\underset{\sim}{e}_i \otimes \underset{\sim}{e}_l \otimes \underset{\sim}{e}_n, \\ \Rightarrow \qquad (\mathbf{A}\times\mathbf{B})_{iln} &= \varepsilon_{jml}\,A_{ij}B_{mn}.\end{aligned}$$

This is a third order tensor.

2.8.7 Expressing matrix multiplication as tensors

In order to help with multiplying dyadics, consider the matrix multiplication of two matrices, say

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}.$$

Then, if we were to calculate **AB**, then element by element we find

$$\begin{aligned} (\mathbf{AB})_{11} &= A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31}, \\ (\mathbf{AB})_{12} &= A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32}, \\ (\mathbf{AB})_{13} &= A_{11}B_{13} + A_{12}B_{23} + A_{13}B_{33}, \\ &\text{etc.} \end{aligned}$$

However, if we compare how each element of **AB** is calculated, then we can see the pattern

$$\begin{aligned} (\mathbf{AB})_{ij} &= A_{i1}B_{1j} + A_{i2}B_{2j} + A_{i3}B_{3j}, \\ &= A_{im}B_{mj}. \end{aligned}$$

Thus, if we multiply two second-order tensors, and the second index of the first tensor is equal to the first index of the second tensor, then this operation is simply equivalent to matrix multiplication.

In the above example, the second index in the first tensor of A_{im}, i.e., m, is equal to the first index of the second tensor of B_{mj}, i.e., m.

Therefore, $(\mathbf{AB})_{ij}$ can be found by multiplying the elements A_{im} by B_{mj}.

Note:

1. Similarly, it can be shown that matrix multiplication occurs when

$$A^{im}B^{mj}, \quad A^i_m B^m_j, \quad A_{im}B^{mj}, \quad \text{etc.}$$

2. This idea of matrix multiplication can be extended to other order tensors, e.g., from Exercise 2.24, we had

$$x^i = \alpha^i_j X^j \Rightarrow \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}.$$

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Exercise 2.27:

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ -3 & 2 & -2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 2 & -1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

Find $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$.

✠

Answer:

□

Summary

In this lecture, we ...

- considered an example on direction cosines
- considered more operations on dyads

Coming up

In the next lecture, we ...

- introduce the double dot product of two dyadics
- introduce the concept of polyads

Homework Exercise 2.11:

1. Let $\underset{\sim}{u} = (1, 2, 3)$, $\underset{\sim}{v} = (3, 1, 2)$ and $\underset{\sim}{w} = (3, 2, 1)$. Find $\mathbf{A}$ and $\mathbf{B}$ if

$$\mathbf{A} = \underset{\sim}{u} \otimes \underset{\sim}{v}, \qquad \mathbf{B} = \underset{\sim}{w} \otimes \underset{\sim}{w}.$$

   Further, find $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$.

2. Consider the two Cartesian coordinate systems with the relationship

$$\bar{\underset{\sim}{e}}_1 = \underset{\sim}{e}_1 - \underset{\sim}{e}_2, \qquad \bar{\underset{\sim}{e}}_2 = \underset{\sim}{e}_2 - \underset{\sim}{e}_3, \qquad \bar{\underset{\sim}{e}}_3 = \underset{\sim}{e}_3 - \underset{\sim}{e}_1.$$

   Find the direction cosines α^i_j and $\bar{\alpha}^i_j$.