# Review

In the previous lecture, we ...

- examine the meaning of physical components
- considered the physical components of higher order tensors

# Aims

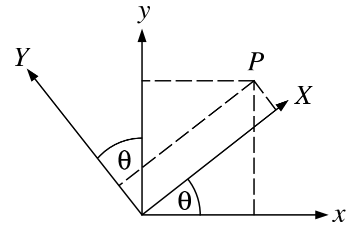
In this lecture, we will ...

- introduce direction cosines
- introduce dyads



## 2.7 Rotation of coordinate axis

Consider two sets of rectangular Cartesian coordinate systems in a plane, namely $(x, y)$ and $(X, Y)$, both centered at the same origin $O$.




The coordinate system $(X, Y)$ is obtained by rotating $Ox$ and $Oy$ through an angle $\theta$, in the counter-clockwise direction.

This transformation of axes is called a *rotation*.



Let $P$ be a point in the $\mathbb{R}^2$ space. In the coordinate system $(x, y)$, $P$ can be expressed in terms of $x$ and $y$, while in the coordinate system $(X, Y)$, $P$ can be expressed in terms of $X$ and $Y$.

Clearly, as $P$ is at the same point in $\mathbb{R}^2$, then there must be a relationship between the two coordinate systems.

Using geometrical arguments, we can write

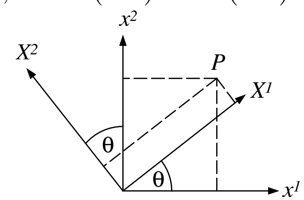
$$\begin{aligned} x &= X\cos\theta - Y\sin\theta, \\ y &= X\sin\theta + Y\cos\theta, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} X &= x\cos\theta + y\sin\theta, \\ Y &= -x\sin\theta + y\cos\theta. \end{aligned} \tag{2.2}$$



If we replace $(x, y)$ and $(X, Y)$ with $(x^1, x^2)$ and $(X^1, X^2)$, respectively, and introduce the notation that $\angle_{X,Y}$ denotes the angle between axes $X$ and $Y$, then (2.1) and (2.2) become




$$\begin{aligned} x^1 &= X^1\cos(\angle_{x^1,X^1}) + X^2\cos(\angle_{x^1,X^2}), \\ x^2 &= X^1\cos(\angle_{x^2,X^1}) + X^2\cos(\angle_{x^2,X^2}), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} X^1 &= x^1\cos(\angle_{X^1,x^1}) + x^2\cos(\angle_{X^1,x^2}), \\ X^2 &= x^1\cos(\angle_{X^2,x^1}) + x^2\cos(\angle_{X^2,x^2}). \end{aligned} \tag{2.4}$$

**Note:**

In (2.3) and (2.4), we have made use of the trigonometric identity

$$\cos(\theta + \pi/2) = \cos\theta\cos\pi/2 - \sin\theta\sin\pi/2 = -\sin\theta.$$

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Upon using index notation, equations (2.3) and (2.4) can be written as

$$\begin{aligned} x^i &= \alpha^i_j\, X^j, \\ X^i &= \bar{\alpha}^i_j\, x^j, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \alpha^i_j &= \cos(\angle x^i, X^j) = \underset{\sim}{e}_i \cdot \underset{\sim}{\bar{e}}_j, \\ \bar{\alpha}^i_j &= \cos(\angle X^i, x^j) = \underset{\sim}{\bar{e}}_i \cdot \underset{\sim}{e}_j \end{aligned}$$

denote the *direction cosines*.



**Note:**

1. In the above, the bar "$-$" does not effect the summation convention. For example,

   $$X^i = \bar{\alpha}^i_j x^j = \bar{\alpha}^i_1 x^1 + \bar{\alpha}^i_2 x^2 + \bar{\alpha}^i_3 x^3.$$

   The bar only refers to which coordinate system you are considering.

2. On comparing (2.5) with Definition 2.4, we find

   $$X^i = \bar{\alpha}^i_j x^j \qquad \text{and} \qquad \bar{A}^i(\underset{\sim}{X}) = A^j(\underset{\sim}{x})\frac{\partial X^i}{\partial x^j}.$$

   This implies that

   $$\bar{\alpha}^i_j = \frac{\partial X^i}{\partial x^j}. \qquad \left(\text{Similarly, } \alpha^i_j = \frac{\partial x^i}{\partial X^j}\right)$$

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### 2.7.1 Orthogonality of direction cosines

Consider the two rectangular Cartesian coordinate systems $(x^1, x^2, x^3)$ and $(X^1, X^2, X^3)$. We know that the set of orthogonal unit vectors $(\underset{\sim}{\bar{e}}_1, \underset{\sim}{\bar{e}}_2, \underset{\sim}{\bar{e}}_3)$ satisfy

$$\underset{\sim}{\bar{e}}_i \cdot \underset{\sim}{\bar{e}}_j = \delta_{ij},$$

and we can write

$$\underset{\sim}{\bar{e}}_i = \bar{\alpha}^m_i \underset{\sim}{e}_m, \qquad \text{and} \qquad \underset{\sim}{\bar{e}}_j = \bar{\alpha}^n_j \underset{\sim}{e}_n.$$

Thus,

$$\begin{aligned} \delta_{ij} &= (\bar{\alpha}^m_i \underset{\sim}{e}_m) \cdot (\bar{\alpha}^n_j \underset{\sim}{e}_n), \\ &= \bar{\alpha}^m_i \bar{\alpha}^n_j (\underset{\sim}{e}_m \cdot \underset{\sim}{e}_n), \\ &= \bar{\alpha}^n_i \bar{\alpha}^n_j. \end{aligned}$$



Thus,

$$\bar{\alpha}^n_i \bar{\alpha}^n_j = \delta_{ij},$$

and similarly, we can show

$$\alpha^n_i \alpha^n_j = \delta_{ij}.$$

These two conditions are called the *orthogonality conditions*.

**Note:**

1. If the direction cosines satisfy the orthogonality conditions, then it means that the lengths of any vectors, and the angles between them, are preserved upon the transformation of the coordinates.

**Note:**

2. For $\alpha_i^n \alpha_j^n = \delta_{ij}$, if $i = j$, then

$$(\alpha_i^1)^2 + (\alpha_i^2)^2 + (\alpha_i^3)^2 = 1.$$

If $i \neq j$, then

$$\alpha_i^1 \alpha_j^1 + \alpha_i^2 \alpha_j^2 + \alpha_i^3 \alpha_j^3 = 0.$$

3. Similar results hold for $\bar{\alpha}_i^n \bar{\alpha}_j^n = \delta_{ij}$.

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**Exercise 2.24:**

The rotated coordinate system $X^i$ makes the following angles with the initial coordinate system $x^i$, namely

| | $X^1$ | $X^2$ | $X^3$ |
|---|---|---|---|
| $x^1$ | 90° | 45° | 135° |
| $x^2$ | 45° | 60° | 60° |
| $x^3$ | 45° | 120° | 120° |

1. Determine the matrix $\alpha_j^i$ of direction cosines.
2. Point $P$ has the coordinates $(0, 1, -1)$ in the $X^i$ system. What are the coordinates in the $x^i$ system?
3. Draw the $X^i$ and $x^i$ systems on the same diagram. Note the position of $P$ in relation to both coordinate systems.

✠



**Answer:**

□



## 2.8 Dyads and dyadics

If $\underset{\sim}{u} = u^i \underset{\sim}{e}_i$ and $\underset{\sim}{v} = v^j \underset{\sim}{e}_j$ are two vectors, then a **dyad** is the *intermediate vector product* of the two vectors $\underset{\sim}{u} \otimes \underset{\sim}{v}$ (sometimes denoted by $\underset{\sim}{u}\underset{\sim}{v}$), such that

$$\underset{\sim}{u} \otimes \underset{\sim}{v} = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} (v^1 \ v^2 \ v^3) = \begin{pmatrix} u^1v^1 & u^1v^2 & u^1v^3 \\ u^2v^1 & u^2v^2 & u^2v^3 \\ u^3v^1 & u^3v^2 & u^3v^3 \end{pmatrix}.$$

A **dyadic** $\mathbf{D}$ is a second order tensor, and may always be written as a finite sum of dyads, i.e.,

$$\mathbf{D} = \underset{\sim}{u}_1 \otimes \underset{\sim}{v}_1 + \underset{\sim}{u}_2 \otimes \underset{\sim}{v}_2 + \underset{\sim}{u}_3 \otimes \underset{\sim}{v}_3 + \cdots + \underset{\sim}{u}_n \otimes \underset{\sim}{v}_n.$$

We can perform the usual sort of operations on dyads.



### 2.8.1 The (vector · dyadic) dot product

Let $\underset{\sim}{a}_i$ and $\underset{\sim}{b}_j$ be vectors such that the dyadic $\mathbf{D}$ is given by

$$\mathbf{D} = \underset{\sim}{a}_1 \otimes \underset{\sim}{b}_1 + \underset{\sim}{a}_2 \otimes \underset{\sim}{b}_2 + \underset{\sim}{a}_3 \otimes \underset{\sim}{b}_3 + \cdots + \underset{\sim}{a}_n \otimes \underset{\sim}{b}_n.$$

If $\underset{\sim}{u}$ is a vector, then the dot products $\underset{\sim}{u} \cdot \mathbf{D}$ and $\mathbf{D} \cdot \underset{\sim}{u}$ are the vectors defined by

$$\underset{\sim}{u} \cdot \mathbf{D} = (\underset{\sim}{u} \cdot \underset{\sim}{a}_1)\underset{\sim}{b}_1 + (\underset{\sim}{u} \cdot \underset{\sim}{a}_2)\underset{\sim}{b}_2 + (\underset{\sim}{u} \cdot \underset{\sim}{a}_3)\underset{\sim}{b}_3 + \cdots + (\underset{\sim}{u} \cdot \underset{\sim}{a}_n)\underset{\sim}{b}_n = \underset{\sim}{v},$$
$$\mathbf{D} \cdot \underset{\sim}{u} = \underset{\sim}{a}_1(\underset{\sim}{b}_1 \cdot \underset{\sim}{u}) + \underset{\sim}{a}_2(\underset{\sim}{b}_2 \cdot \underset{\sim}{u}) + \underset{\sim}{a}_3(\underset{\sim}{b}_3 \cdot \underset{\sim}{u}) + \cdots + \underset{\sim}{a}_n(\underset{\sim}{b}_n \cdot \underset{\sim}{u}) = \underset{\sim}{w}.$$

**Note:**

The (vector · dyadic) dot products produce a new vector.

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### 2.8.2 The (vector × dyadic) cross product

Let $\underset{\sim}{a}_i$ and $\underset{\sim}{b}_j$ be vectors such that the dyadic $\mathbf{D}$ is given by

$$\mathbf{D} = \underset{\sim}{a}_1 \otimes \underset{\sim}{b}_1 + \underset{\sim}{a}_2 \otimes \underset{\sim}{b}_2 + \underset{\sim}{a}_3 \otimes \underset{\sim}{b}_3 + \cdots + \underset{\sim}{a}_n \otimes \underset{\sim}{b}_n.$$

If $\underset{\sim}{u}$ is a vector, then the cross products $\underset{\sim}{u} \times \mathbf{D}$ and $\mathbf{D} \times \underset{\sim}{u}$ are the dyadics defined by

$$\underset{\sim}{u} \times \mathbf{D} = (\underset{\sim}{u} \times \underset{\sim}{a}_1) \otimes \underset{\sim}{b}_1 + (\underset{\sim}{u} \times \underset{\sim}{a}_2) \otimes \underset{\sim}{b}_2 + \cdots + (\underset{\sim}{u} \times \underset{\sim}{a}_n) \otimes \underset{\sim}{b}_n = \mathbf{F},$$
$$\mathbf{D} \times \underset{\sim}{u} = \underset{\sim}{a}_1 \otimes (\underset{\sim}{b}_1 \times \underset{\sim}{u}) + \underset{\sim}{a}_2 \otimes (\underset{\sim}{b}_2 \times \underset{\sim}{u}) + \cdots + \underset{\sim}{a}_n \otimes (\underset{\sim}{b}_n \times \underset{\sim}{u}) = \mathbf{G}.$$

**Note:**

The (vector × dyadic) cross products produce a new dyadic.

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**Example 2.24:**
If $\underset{\sim}{u} = (1, 0, 2)$ and $\underset{\sim}{v} = (-1, 1, 0)$, then find the dyads $\underset{\sim}{u} \otimes \underset{\sim}{v}$ and $\underset{\sim}{v} \otimes \underset{\sim}{u}$.

**Answer:**

$$\underset{\sim}{u} \otimes \underset{\sim}{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} (-1\ 1\ 0) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ -2 & 2 & 0 \end{pmatrix},$$

and

$$\underset{\sim}{v} \otimes \underset{\sim}{u} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} (1\ 0\ 2) = \begin{pmatrix} -1 & 0 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

□



**Exercise 2.25:**

Let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be as given in Example 2.24, and let $\underset{\sim}{w} = (3, 1, 2)$. Find

1. $\mathbf{D} = \underset{\sim}{u} \otimes \underset{\sim}{v} + \underset{\sim}{v} \otimes \underset{\sim}{u}$.
2. $\underset{\sim}{w} \cdot \mathbf{D}$.
3. $\mathbf{D} \times \underset{\sim}{w}$.

□

**Answer:**



□



## Summary

In this lecture, we ...

- introduced direction cosines
- introduced dyads

## Coming up

In the next lecture, we ...

- consider an example on direction cosines
- consider more operations on dyads

**Homework Exercise 2.10:**

1. Given that

$$\mathbf{T} = \begin{pmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{pmatrix}$$

is relative to the bases $\underset{\sim}{e}_i$, and

$$\begin{aligned} \bar{\underset{\sim}{e}}_1 &= \frac{1}{\sqrt{5}}\, \underset{\sim}{e}_2 + \frac{2}{\sqrt{5}}\, \underset{\sim}{e}_3, \\ \bar{\underset{\sim}{e}}_2 &= \underset{\sim}{e}_1, \\ \bar{\underset{\sim}{e}}_3 &= \beta_1\, \underset{\sim}{e}_1 + \beta_2\, \underset{\sim}{e}_2 + \beta_3\, \underset{\sim}{e}_3, \end{aligned}$$

then determine the $\beta_i$'s. Hence, or otherwise, find the rotation matrix $\bar{\alpha}^i_j$.
(Hint: use the relation $\bar{\alpha}^m_i\, \bar{\alpha}^m_j = \delta_{ij}$)