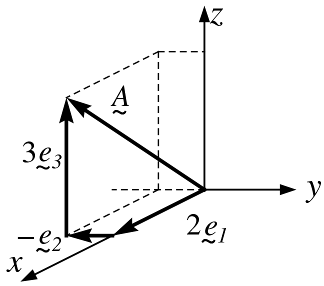
# MATH312 Tutorial Solutions
# Autumn 2008
# Week 5

**Question 1:**

1. $\underset{\sim}{A} = 2\underset{\sim}{e}_1 - \underset{\sim}{e}_2 + 3\underset{\sim}{e}_3$ in Cartesian coordinates.

   (a)



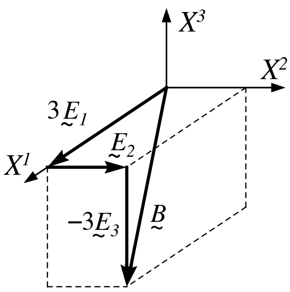

   (b) Vector $\underset{\sim}{A}$ starts at the origin, and moves 2 units in the $x$-direction, then $-1$ unit in the $y$-direction, and finally 3 units in the $z$-direction. Thus, the vector's end point is $(2, -1, 3)$.

   (c) As vector $\underset{\sim}{A}$ is given in Cartesian coordinates and in terms of orthonormal base vectors, then the physical components are exactly equal to the contravariant components, namely

$$A^{(1)} = A_{(1)} = A^1 = 2, \qquad A^{(2)} = A_{(2)} = A^2 = -1, \qquad A^{(3)} = A_{(3)} = A^3 = 3.$$

2. $\underset{\sim}{B} = 3\underset{\sim}{E}_1 + \underset{\sim}{E}_2 - 3\underset{\sim}{E}_3$ in an orthogonal curvilinear coordinate system.

   (a) Upon rotating our view of the orthogonal curvilinear coordinates to follow the Cartesian coordinates, we find




   (b) Vector $\underset{\sim}{B}$ starts at the origin, and moves 2 lengths of vector $\underset{\sim}{E}_1$ in the $X^1$-direction, then 1 length of vector $\underset{\sim}{E}_2$ in the $X^2$-direction, and finally $-3$ lengths of vector $\underset{\sim}{E}_3$ in $X^3$-direction. Thus, the vector's end point is $(3|\underset{\sim}{E}_1|, |\underset{\sim}{E}_2|, -3|\underset{\sim}{E}_3|)$.

(c) The contravariant physical components of $\underset{\sim}{B}$ are the physical distances (or number of unit lengths) the vector moves in the $X^1$, $X^2$ and $X^3$ directions. Thus,

$$\begin{aligned}\underset{\sim}{B} &= 3\underset{\sim}{E}_1 + \underset{\sim}{E}_2 - 3\underset{\sim}{E}_3,\\ &= 3|\underset{\sim}{E}_1|\frac{\underset{\sim}{E}_1}{|\underset{\sim}{E}_1|} + |\underset{\sim}{E}_2|\frac{\underset{\sim}{E}_2}{|\underset{\sim}{E}_2|} - 3|\underset{\sim}{E}_3|\frac{\underset{\sim}{E}_3}{|\underset{\sim}{E}_3|},\end{aligned}$$

i.e.,

$$B^{(1)} = 3|\underset{\sim}{E}_1|, \qquad B^{(2)} = |\underset{\sim}{E}_2|, \qquad B^{(3)} = -3|\underset{\sim}{E}_3|.$$

To find the covariant physical components, we need the gradient basis vectors $(\underset{\sim}{E}^1, \underset{\sim}{E}^2, \underset{\sim}{E}^3)$ and the covariant components $(B_1, B_2, B_3)$. Now, we have shown in lectures that

$$B_i = B^{\underline{i}}|\underset{\sim}{E}_{\underline{i}}|^2,$$

which yields

$$B_1 = B^1|\underset{\sim}{E}_1|^2 = 3|\underset{\sim}{E}_1|^2, \quad B_2 = B^2|\underset{\sim}{E}_2|^2 = |\underset{\sim}{E}_2|^2, \quad B_3 = B^3|\underset{\sim}{E}_3|^2 = -3|\underset{\sim}{E}_3|^2.$$

Thus, from the formula

$$B_{(i)} = B_{\underline{i}}|\underset{\sim}{E}^{\underline{i}}|,$$

we find

$$B_{(1)} = 3|\underset{\sim}{E}_1|^2|\underset{\sim}{E}^1|, \qquad B_{(2)} = |\underset{\sim}{E}_2|^2|\underset{\sim}{E}^2|, \qquad B_{(3)} = -3|\underset{\sim}{E}_3|^2|\underset{\sim}{E}^3|.$$

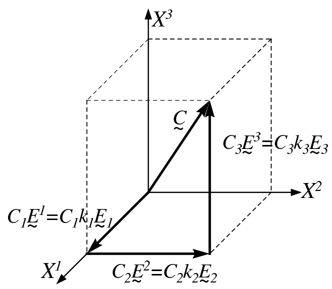
3. $\underset{\sim}{C} = C_1\underset{\sim}{E}^1 + C_2\underset{\sim}{E}^2 + C_3\underset{\sim}{E}^3$ in an orthogonal curvilinear coordinate system, where $(\underset{\sim}{E}^1, \underset{\sim}{E}^2, \underset{\sim}{E}^3)$ are gradient basis vectors.

(a) As the curvilinear coordinate system is orthogonal, then (from performing cross products) we know

$$\underset{\sim}{E}^1 = k_1\underset{\sim}{E}_1, \qquad \underset{\sim}{E}^2 = k_2\underset{\sim}{E}_2, \qquad \underset{\sim}{E}^3 = k_3\underset{\sim}{E}_3,$$

where $k_1, k_2, k_3$ are three specific non-zero scalars and $(\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3)$ are the tangential basis vectors. The values of $k_1, k_2$ and $k_3$ are determined by ensuring that the tangential and gradient basis vectors form a reciprocal basis, and is related to the scalar triple product $V = \underset{\sim}{E}_1 \cdot (\underset{\sim}{E}_2 \times \underset{\sim}{E}_3)$.

(b) Upon rotating our view of the orthogonal curvilinear coordinates to follow the Cartesian coordinates, we find



(c) Vector $\underset{\sim}{C}$ starts at the origin, and moves $C_1$ lengths of vector $\underset{\sim}{E}^1$ in the direction of the $\underset{\sim}{E}^1$ vector, which due to orthogonality, here $\underset{\sim}{E}^1$ is in $X^1$-direction. Then $\underset{\sim}{C}$ moves $C_2$ lengths of vector $\underset{\sim}{E}^2$ in the direction of $\underset{\sim}{E}^2$, which again due to orthogonality is in the $X^2$-direction, and finally $C_3$ units of vector $\underset{\sim}{E}^3$ in $X^3$-direction. Thus, the vector's end point is $(C_1|\underset{\sim}{E}^1|, C_2|\underset{\sim}{E}^2|, C_3|\underset{\sim}{E}^3|)$. Alternatively, you could express the movement of $\underset{\sim}{C}$ in terms of the tangential basis vectors.

(d) The physical components of $\underset{\sim}{C}$ are the physical distances (or number of unit lengths) the vector moves in the $X_1$, $X_2$ and $X_3$ directions. Thus,

$$\begin{aligned}\underset{\sim}{C} &= C_1\underset{\sim}{E}^1 + C_2\underset{\sim}{E}^2 + C_3\underset{\sim}{E}^3 = C_1k_1\underset{\sim}{E}_1 + C_2k_2\underset{\sim}{E}_2 + C_3k_3\underset{\sim}{E}_3,\\ &= C_1|\underset{\sim}{E}^1|\frac{\underset{\sim}{E}^1}{|\underset{\sim}{E}^1|} + C_2|\underset{\sim}{E}^2|\frac{\underset{\sim}{E}^2}{|\underset{\sim}{E}^2|} + C_3|\underset{\sim}{E}^3|\frac{\underset{\sim}{E}^3}{|\underset{\sim}{E}^3|}, \text{ or,}\\ &= C_1k_1|\underset{\sim}{E}_1|\frac{\underset{\sim}{E}_1}{|\underset{\sim}{E}_1|} + C_2k_1|\underset{\sim}{E}_2|\frac{\underset{\sim}{E}_2}{|\underset{\sim}{E}_2|} + C_3k_3|\underset{\sim}{E}_3|\frac{\underset{\sim}{E}_3}{|\underset{\sim}{E}_3|},\end{aligned}$$

i.e.,

$$C_{(1)} = C_1|\underset{\sim}{E}^1|, \quad C_{(2)} = C_2|\underset{\sim}{E}^2|, \quad C_{(3)} = C_3|\underset{\sim}{E}^3|,$$

and

$$C^{(1)} = C_1k_1|\underset{\sim}{E}_1|, \qquad C^{(2)} = C_2k_2|\underset{\sim}{E}_2|, \qquad C^{(3)} = C_3k_3|\underset{\sim}{E}_3|.$$

Further, this implies

$$C^1 = C_1k_1, \qquad C^2 = C_2k_2, \qquad C^3 = C_3k_3.$$

4. Consider the modified cylindrical polar coordinates

$$z^1 = 3X^1\cos X^2, \qquad z^2 = 3X^1\sin X^2, \qquad z^3 = 2X^3.$$

(a) Let the position vector be given by $\underset{\sim}{r} = z^1\underset{\sim}{e}_1 + z^2\underset{\sim}{e}_2 + z^3\underset{\sim}{e}_3$, so that

$$\underset{\sim}{r} = 3X^1\cos X^2\underset{\sim}{e}_1 + 3X^1\sin X^2\underset{\sim}{e}_2 + 2X^3\underset{\sim}{e}_3.$$

Thus, if $\underset{\sim}{E}_i = \dfrac{\partial\underset{\sim}{r}}{\partial X_i}$, then

$$\begin{aligned}\underset{\sim}{E}_1 &= 3\cos X^2\underset{\sim}{e}_1 + 3\sin X^2\underset{\sim}{e}_2,\\ \underset{\sim}{E}_2 &= -3X^1\sin X^2\underset{\sim}{e}_1 + 3X^1\cos X^2\underset{\sim}{e}_2,\\ \underset{\sim}{E}_3 &= 2\underset{\sim}{e}_3.\end{aligned}$$

To find the gradient basis vectors, we know

$$\underset{\sim}{E}^1 = \frac{1}{V}(\underset{\sim}{E}_2\times\underset{\sim}{E}_3), \qquad \underset{\sim}{E}^2 = \frac{1}{V}(\underset{\sim}{E}_3\times\underset{\sim}{E}_1), \qquad \underset{\sim}{E}^3 = \frac{1}{V}(\underset{\sim}{E}_1\times\underset{\sim}{E}_2),$$

where $V = \underset{\sim}{E}_1\cdot(\underset{\sim}{E}_2\times\underset{\sim}{E}_3) \neq 0$ is the triple scalar product and represents the volume of the parallelepiped having the basis vectors for its sides. Note it is important that the order of indices remain in the same cycle (in a similar manner to the permutation symbol $\varepsilon_{ijk}$). If the order is not maintained, then $V$ will be different. Thus, we first need to find $V$, i.e.,

$$\begin{aligned} V = \underset{\sim}{E}_1 \cdot (\underset{\sim}{E}_2 \times \underset{\sim}{E}_3) &= \begin{vmatrix} 3\cos X^2 & 3\sin X^2 & 0 \\ -3X^1 \sin X^2 & 3X^1 \cos X^2 & 0 \\ 0 & 0 & 2 \end{vmatrix}, \\ &= 2(9X^1 \cos^2 X^2 + 9X^2 \sin^2 X^2), \\ &= 18X^1. \end{aligned}$$

Hence, we now consider

$$\begin{aligned} \underset{\sim}{E}^1 = \frac{1}{V}(\underset{\sim}{E}_2 \times \underset{\sim}{E}_3) &= \frac{1}{18X^1} \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ -3X^1 \sin X^2 & 3X^1 \cos X^2 & 0 \\ 0 & 0 & 2 \end{vmatrix}, \\ &= \frac{1}{18X^1}(6X^1 \cos X^2 \underset{\sim}{e}_1 + 6X^1 \sin X^2 \underset{\sim}{e}_2), \\ &= \frac{1}{3} \cos X^2 \underset{\sim}{e}_1 + \frac{1}{3} \sin X^2 \underset{\sim}{e}_2, \\ &= \frac{1}{9} \underset{\sim}{E}_1. \end{aligned}$$

Similarly, noting the cyclic order of the indices, we consider

$$\begin{aligned} \underset{\sim}{E}^2 = \frac{1}{V}(\underset{\sim}{E}_3 \times \underset{\sim}{E}_1) &= \frac{1}{18X^1} \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ 0 & 0 & 2 \\ 3\cos X^2 & 3\sin X^2 & 0 \end{vmatrix}, \\ &= \frac{1}{18X^1}(-6\sin X^2 \underset{\sim}{e}_1 + 6\cos X^2 \underset{\sim}{e}_2), \\ &= -\frac{1}{3X^1} \sin X^2 \underset{\sim}{e}_1 + \frac{1}{3X^1} \cos X^2 \underset{\sim}{e}_2, \\ &= \frac{1}{9(X^1)^2} \underset{\sim}{E}_2. \end{aligned}$$

Finally, to find $\underset{\sim}{E}^3$, we consider

$$\begin{aligned} \underset{\sim}{E}^3 = \frac{1}{V}(\underset{\sim}{E}_1 \times \underset{\sim}{E}_2) &= \frac{1}{18X^1} \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ 3\cos X^2 & 3\sin X^2 & 0 \\ -3X^1 \sin X^2 & 3X^1 \cos X^2 & 0 \end{vmatrix}, \\ &= \frac{1}{18X^1}(9X^1 \underset{\sim}{e}_3), \\ &= \frac{1}{2} \underset{\sim}{e}_3, \\ &= \frac{1}{4} \underset{\sim}{E}_3. \end{aligned}$$

As a check, we should ensure that the tangential and gradient basis vectors are reciprocal, i.e., $\underset{\sim}{E}_i \cdot \underset{\sim}{E}^j = \delta_i^j$. Thus,

$$
\begin{aligned}
(i = j = 1) \quad &\Rightarrow \quad 1 = \underset{\sim}{E}_1 \cdot \underset{\sim}{E}^1 = (3\cos X^2, 3\sin X^2, 0) \cdot \left(\frac{1}{3}\cos X^2, \frac{1}{3}\sin X^2, 0\right), \\
&\qquad\quad = 1. \quad \checkmark \\
(i = 1, j = 2) \quad &\Rightarrow \quad 0 = \underset{\sim}{E}_1 \cdot \underset{\sim}{E}^2 = (3\cos X^2, 3\sin X^2, 0) \cdot \left(-\frac{1}{3X^1}\sin X^2, \frac{1}{3X^1}\cos X^2, 0\right), \\
&\qquad\quad = 0. \quad \checkmark \\
(i = 1, j = 3) \quad &\Rightarrow \quad 0 = \underset{\sim}{E}_1 \cdot \underset{\sim}{E}^3 = (3\cos X^2, 3\sin X^2, 0) \cdot \left(0, 0, \frac{1}{2}\right), \\
&\qquad\quad = 0. \quad \checkmark
\end{aligned}
$$

etc.

Note, if we included arbitrary scalars $k_1$, $k_2$ and $k_3$ in the definition of the gradient basis vectors, then we would find that upon checking the basis vectors satisfy the reciprocal condition the arbitrary scalars must equal 1.

(b) We have previously found that

$$B^{(1)} = 3|\underset{\sim}{E}_1|, \qquad B^{(2)} = |\underset{\sim}{E}_2|, \qquad B^{(3)} = -3|\underset{\sim}{E}_3|,$$

and

$$B_{(1)} = 3|\underset{\sim}{E}_1|^2|\underset{\sim}{E}^1|, \qquad B_{(2)} = |\underset{\sim}{E}_2|^2|\underset{\sim}{E}^2|, \qquad B_{(3)} = -3|\underset{\sim}{E}_3|^2|\underset{\sim}{E}^3|,$$

which implies

$$
\begin{aligned}
B_1 &= 3|\underset{\sim}{E}_1|^2, \\
B_2 &= |\underset{\sim}{E}_2|^2, \\
B_3 &= -3|\underset{\sim}{E}_3|^2.
\end{aligned}
$$

Hence, given the tangential basis vectors above, we find

$$
\begin{aligned}
B^{(1)} &= 3\sqrt{9\cos^2 X^2 + 9\sin^2 X^2 + 0^2}, \\
&= 9. \\
B^{(2)} &= \sqrt{9(X^1)^2\sin^2 X^2 + 9(X^1)^2\cos^2 X^2 + 0^2}, \\
&= 3X^1. \\
B^{(3)} &= -3\sqrt{0^2 + 0^2 + 4}, \\
&= -6.
\end{aligned}
$$

And,

$$B_{(1)} = 3(9\cos^2 X^2 + 9\sin^2 X^2 + 0^2)\sqrt{\frac{1}{9}\cos^2 X^2 + \frac{1}{9}\sin^2 X^2 + 0^2}$$
$$= 9.$$
$$B_{(2)} = (9(X^1)^2\sin^2 X^2 + 9(X^1)^2\cos^2 X^2 + 0^2)\sqrt{\frac{1}{9(X^1)^2}\sin^2 X^2 + \frac{1}{3(X^1)^2}\cos^2 X^2 + 0^2},$$
$$= 3X^1.$$
$$B_{(3)} = -3(0^2 + 0^2 + 4)\sqrt{0^2 + 0^2\frac{1}{4}},$$
$$= -6.$$

Note: For a specified value of $i$, the physical components are equal, i.e., $B^{(i)} = B_{(i)}$. This is due to the fact that the respective gradient basis vectors are in the direction of the corresponding tangential basis vectors. Or, in other words, if $k_i$ is a scalar then $\underset{\sim}{E}_i = k_i\underset{\sim}{E}^i$, so that if you find the unit vector in the direction of $\underset{\sim}{E}_i$, then this is identically equal to the unit vector in the direction of $\underset{\sim}{E}^i$. Hence, the number of units you move in the appropriate directions will be the same.

Similarly, for the physical components of $\underset{\sim}{C}$, we found

$$C_{(1)} = C_1|\underset{\sim}{E}^1|, \quad C_{(2)} = C_2|\underset{\sim}{E}^2|, \quad C_{(3)} = C_3|\underset{\sim}{E}^3|,$$

and

$$C^{(1)} = C_1k_1|\underset{\sim}{E}_1|, \qquad C^{(2)} = C_2k_2|\underset{\sim}{E}_2|, \qquad C^{(3)} = C_3k_3|\underset{\sim}{E}_3|,$$

where

$$C^1 = C_1k_1, \qquad C^2 = C_2k_2, \qquad C^3 = C_3k_3.$$

The scalars $k_1$, $k_2$ and $k_3$ need to be determined by the reciprocal basis condition. However, to begin, we find

$$C_{(1)} = C_1\sqrt{\frac{1}{9}\cos^2 X^2 + \frac{1}{9}\sin^2 X^2 + 0^2},$$
$$= \frac{C_1}{3}.$$
$$C_{(2)} = C_2\sqrt{\frac{1}{9(X^1)^2}\sin^2 X^2 + \frac{1}{9(X^1)^2}\cos^2 X^2},$$
$$= \frac{C_2}{3X^1}.$$
$$C_{(3)} = C_3\sqrt{0^2 + 0^2\frac{1}{4}},$$
$$= \frac{C_3}{2}.$$

And,

$$\begin{aligned}
C^{(1)} &= C_1 k_1 \sqrt{9\cos^2 X^2 + 9\sin^2 X^2 + 0^2}, \\
&= 3C_1 k_1. \\
C^{(2)} &= C_2 k_2 \sqrt{9(X^1)^2 \sin^2 X^2 + 9(X^1)^2 \cos^2 X^2 + 0^2}, \\
&= 3C_2 k_2 X^1. \\
C^{(3)} &= C_3 k_3 \sqrt{0^2 + 0^2 + 4}, \\
&= 2C_3 k_3.
\end{aligned}$$

Now, to find the values of the scalars $k_1$, $k_2$ and $k_3$, we consider the reciprocal basis condition $\underset{\sim}{E}_i \cdot \underset{\sim}{E}^j = \delta_i^j$. Thus,

$$\begin{aligned}
(i = j = 1) \quad &\Rightarrow \quad 1 = \underset{\sim}{E}_1 \cdot \underset{\sim}{E}^1 = \underset{\sim}{E}_1 \cdot (k_1 \underset{\sim}{E}_1) = k_1 |\underset{\sim}{E}_1|^2, \\
&\qquad\; = k_1 (9\cos^2 X^2 + 9\sin^2 X^2 + 0^2), \\
&\Rightarrow \quad k_2 = \frac{1}{9}. \\
(i = j = 2) \quad &\Rightarrow \quad 1 = \underset{\sim}{E}_2 \cdot \underset{\sim}{E}^2 = \underset{\sim}{E}_2 \cdot (k_2 \underset{\sim}{E}_2) = k_2 |\underset{\sim}{E}_2|^2, \\
&\qquad\; = k_2 (9(X^1)^2 \sin^2 X^2 + 9(X^1)^2 \cos^2 X^2 + 0^2), \\
&\Rightarrow \quad k_2 = \frac{1}{9(X^1)^2}. \\
(i = j = 3) \quad &\Rightarrow \quad 1 = \underset{\sim}{E}_3 \cdot \underset{\sim}{E}^3 = \underset{\sim}{E}_3 \cdot (k_3 \underset{\sim}{E}_3) = k_3 |\underset{\sim}{E}_3|^2, \\
&\qquad\; = k_3 (0^2 + 0^2 + 4), \\
&\Rightarrow \quad k_3 = \frac{1}{4}.
\end{aligned}$$

Note: With these values of $k_1$, $k_2$ and $k_3$ then the respective contravariant and the covariant physical components become identical. This happens for the same reason given in (b). You should also check all the other combinations of values of $i$ and $j$ to ensure that the reciprocal conditions are true.

Further note: In order to check answers, if we assume the contravariant components of $\underset{\sim}{C}$ are identical to the contravariant components of $\underset{\sim}{B}$, i.e.,

$$C^1 = 3, \qquad C^2 = 1, \qquad C^3 = -3,$$

then from

$$C_i = C^{\underline{i}} |\underset{\sim}{E}_{\underline{i}}|^2,$$

we find the covariant components of $\underset{\sim}{C}$ to be

$$\begin{aligned}
C_1 &= C^1 |\underset{\sim}{E}_1|^2 = 3(9\cos^2 X^2 + 9\sin^2 X^2 + 0^2), \\
&= 27, \\
C_2 &= C^2 |\underset{\sim}{E}_2|^2 = (9(X^1)^2 \sin^2 X^2 + 9(X^1)^2 \cos^2 X^2 + 0^2), \\
&= 9(X^1)^2, \\
C_3 &= C^3 |\underset{\sim}{E}_3|^2 = -3(0^2 + 0^2 + 4), \\
&= -12.
\end{aligned}$$

Therefore, in this case, the contravariant and covariant physical components of $\underset{\sim}{C}$ are given by

$$\begin{aligned} C^{(1)} &= \frac{27}{3} = 9, \\ &= C_{(1)} = B_{(1)} = B^{(1)}, \\ C^{(2)} &= \frac{9(X^1)^2}{3X^1} = 3X^1, \\ &= C_{(2)} = B_{(2)} = B^{(2)}, \\ C^{(3)} &= \frac{-12}{2} = -6, \\ &= C_{(3)} = B_{(3)} = B^{(3)}, \end{aligned}$$

as expected.

**Question?**
Let

$$t^i_j = \begin{pmatrix} 1 & a(\underset{\sim}{X}) & b(\underset{\sim}{X}) \\ c(\underset{\sim}{X}) & 1 & 0 \\ d(\underset{\sim}{X}) & 0 & e(\underset{\sim}{X}) \end{pmatrix},$$

where $a(\underset{\sim}{X})$, $b(\underset{\sim}{X})$, $c(\underset{\sim}{X})$, $d(\underset{\sim}{X})$ and $e(\underset{\sim}{X})$ are scalar functions of $\underset{\sim}{X} = (X^1, X^2, X^3)$.

1. If $(X^1, X^2, X^3)$ refer to the usual Cartesian coordinates, then the metric and conjugate metric tensors are

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. Thus, from

$$t^{(i)}_{(j)} = \sqrt{\frac{g_{ii}}{g_{jj}}}\, t^i_j,$$

we find

$$t^{(i)}_{(j)} = \begin{pmatrix} 1 & a(\underset{\sim}{X}) & b(\underset{\sim}{X}) \\ c(\underset{\sim}{X}) & 1 & 0 \\ d(\underset{\sim}{X}) & 0 & e(\underset{\sim}{X}) \end{pmatrix}.$$

Thus, as each element of a physical tensor has the same dimensions, then

$$[a(\underset{\sim}{X})] = [b(\underset{\sim}{X})] = [c(\underset{\sim}{X})] = [d(\underset{\sim}{X})] = [e(\underset{\sim}{X})] = [1] = \text{constant},$$

where $[\cdot]$ denotes the dimensions function. For example,

(a) $a(\underset{\sim}{X}) = 3\frac{\sqrt{(X^1)^2 + (X^2)^2}}{X^3}$,

(b) $b(\underset{\sim}{X}) = 42$,

(c) $c(\underset{\sim}{X}) = \sin\frac{X^1}{X^2}$,

(d) $d(\underset{\sim}{X}) = 0$,

(e) $e(\underset{\sim}{X}) = \mathrm{e}^{(X^1+X^2)^2/(X^3)^2}$,

given that in Cartesian coordinates: $[X^1] = [X^2] = [X^3] = \text{length}$. Please note that these example are not unique.

2. If $(X^1, X^2, X^3)$ refer to the usual cylindrical polar coordinates, then the metric and conjugate metric tensors are

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (X^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{(X^1)^2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. Thus, from

$$t^{(i)}_{(j)} = \sqrt{\frac{g_{ii}}{g_{jj}}}\, t^i_j,$$

we find

$$t^{(\underline{i})}_{(\underline{i})} = t^{\underline{i}}_{\underline{i}}, \qquad t^{(2)}_{(3)} = t^{(3)}_{(2)} = 0,$$

and

$$\begin{aligned} t^{(1)}_{(2)} &= \sqrt{\frac{g_{11}}{g_{22}}} t^1_2 = \sqrt{\frac{1}{(X^1)^2}} a(\underset{\sim}{X}) = \frac{a(\underset{\sim}{X})}{X^1}, \\ t^{(1)}_{(3)} &= \sqrt{\frac{g_{11}}{g_{33}}} t^1_3 = \sqrt{\frac{1}{1}} b(\underset{\sim}{X}) = b(\underset{\sim}{X}), \\ t^{(2)}_{(1)} &= \sqrt{\frac{g_{22}}{g_{11}}} t^2_1 = \sqrt{\frac{(X^1)^2}{1}} c(\underset{\sim}{X}) = X^1 c(\underset{\sim}{X}), \\ t^{(3)}_{(1)} &= \sqrt{\frac{g_{33}}{g_{11}}} t^3_1 = \sqrt{\frac{1}{1}} d(\underset{\sim}{X}) = d(\underset{\sim}{X}), \end{aligned}$$

noting that, in general, $t^i_j \neq t^j_i$ and $t^{(i)}_{(j)} \neq t^{(j)}_{(i)}$. Therefore, the physical tensor $t^{(i)}_{(j)}$ is given by

$$t^{(i)}_{(j)} = \begin{pmatrix} 1 & \frac{a(\underset{\sim}{X})}{X^1} & b(\underset{\sim}{X}) \\ X^1 c(\underset{\sim}{X}) & 1 & 0 \\ d(\underset{\sim}{X}) & 0 & e(\underset{\sim}{X}) \end{pmatrix}.$$

Thus, as each element of a physical tensor has the same dimensions, then

$$\left[\frac{a(\underset{\sim}{X})}{X^1}\right] = [b(\underset{\sim}{X})] = [X^1 c(\underset{\sim}{X})] = [d(\underset{\sim}{X})] = [e(\underset{\sim}{X})] = [1] = \text{constant},$$

so that

(a) $a(\underset{\sim}{X})$ has dimensions of $[X^1] = \text{length}$,

(b) $b(\underset{\sim}{X})$ has dimensions of $[1] = \text{constant}$,

(c) $c(\underset{\sim}{X})$ has dimensions of $\left[\frac{1}{X^1}\right] = \text{length}^{-1}$,

(d) $d(\underset{\sim}{X})$ has dimensions of $[1] = \text{constant}$,

(e) $e(\underset{\sim}{X})$ has dimensions of $[1] = \text{constant}$.

This in turn leads to the examples

(a) $a(\underset{\sim}{X}) = X^1 \cos X^2$,

(b) $b(\underset{\sim}{X}) = 42$,

(c) $c(\underset{\sim}{X}) = \dfrac{X^2}{X^1 + X^3}$,

(d) $d(\underset{\sim}{X}) = 0$,

(e) $e(\underset{\sim}{X}) = \mathrm{e}^{(X^1+X^3)^2/(X^1X^3)}$,

given that $[X^1] = [X^3] =$ length and $[X^2] =$ [angle] $=$ constant. Again, these examples are not unique.

3. If $(X^1, X^2, X^3)$ refer to the usual spherical polar coordinates, then the metric and conjugate metric tensors are

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (X^1)^2 & 0 \\ 0 & 0 & (X^1)^2 \sin^2 X^2 \end{pmatrix}, \qquad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{(X^1)^2} & 0 \\ 0 & 0 & \frac{1}{(X^1)^2 \sin^2 X^2} \end{pmatrix},$$

respectively. Thus, from

$$t^{(i)}_{(j)} = \sqrt{\frac{g_{ii}}{g_{jj}}}\, t^i_j,$$

we find

$$t^{(\underline{i})}_{(\underline{i})} = t^{\underline{i}}_{\underline{i}}, \qquad t^{(2)}_{(3)} = t^{(3)}_{(2)} = 0,$$

and

$$t^{(1)}_{(2)} = \sqrt{\frac{g_{11}}{g_{22}}} t^1_2 = \sqrt{\frac{1}{(X^1)^2}} a(\underset{\sim}{X}) = \frac{a(\underset{\sim}{X})}{X^1},$$

$$t^{(1)}_{(3)} = \sqrt{\frac{g_{11}}{g_{33}}} t^1_3 = \sqrt{\frac{1}{(X^1)^2 \sin^2 X^2}} b(\underset{\sim}{X}) = \frac{b(\underset{\sim}{X})}{X^1 \sin X^2},$$

$$t^{(2)}_{(1)} = \sqrt{\frac{g_{22}}{g_{11}}} t^2_1 = \sqrt{\frac{(X^1)^2}{1}} c(\underset{\sim}{X}) = X^1 c(\underset{\sim}{X}),$$

$$t^{(3)}_{(1)} = \sqrt{\frac{g_{33}}{g_{11}}} t^3_1 = \sqrt{\frac{(X^1)^2 \sin^2 X^2}{1}} d(\underset{\sim}{X}) = X^1 d(\underset{\sim}{X}) \sin X^2,$$

Therefore, the physical tensor $t^{(i)}_{(j)}$ is given by

$$t^{(i)}_{(j)} = \begin{pmatrix} 1 & \frac{a(\underset{\sim}{X})}{X^1} & \frac{b(\underset{\sim}{X})}{X^1 \sin X^2} \\ X^1 c(\underset{\sim}{X}) & 1 & 0 \\ X^1 d(\underset{\sim}{X}) \sin X^2 & 0 & e(\underset{\sim}{X}) \end{pmatrix}.$$

Thus, as each element of a physical tensor has the same dimensions, then

$$\left[\frac{a(\underset{\sim}{X})}{X^1}\right] = \left[\frac{b(\underset{\sim}{X})}{X^1 \sin X^2}\right] = [X^1 c(\underset{\sim}{X})] = [X^1 d(\underset{\sim}{X}) \sin X^2] = [e(\underset{\sim}{X})] = [1] = \text{constant},$$

so that

(a) $a(\underset{\sim}{X})$ has dimensions of $[X^1] =$ length,
(b) $b(\underset{\sim}{X})$ has dimensions of $[X^1 \sin X^2] =$ length,
(c) $c(\underset{\sim}{X})$ has dimensions of $\left[\frac{1}{X^1}\right] = \text{length}^{-1}$,
(d) $d(\underset{\sim}{X})$ has dimensions of $\left[\frac{1}{X^1 \sin X^2}\right] = \text{length}^{-1}$,
(e) $e(\underset{\sim}{X})$ has dimensions of $[1] =$ constant.

This in turn leads to the examples

(a) $a(\underset{\sim}{X}) = X^1 \cos X^2$,
(b) $b(\underset{\sim}{X}) = 42X^1$,
(c) $c(\underset{\sim}{X}) = \dfrac{1}{X^1}$,
(d) $d(\underset{\sim}{X}) = \frac{42}{X^1(\sin X^2 + \cos X^3)}$,
(e) $e(\underset{\sim}{X}) = \mathrm{e}^{X^3/X^2}$,

given that $[X^1] =$ length and $[X^2] = [X^3] = [\text{angle}] =$ constant. Again, these examples are not unique.

4. If $(X^1, X^2, X^3)$ refer to the modified cylindrical polar coordinates

$$z^1 = 3X^1 \cos X^2, \qquad z^2 = 3X^1 \sin X^2, \qquad z^3 = 2X^3,$$

then the metric and conjugate metric tensors are

$$g_{ij} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9(X^1)^2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \qquad g^{ij} = \begin{pmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{9(X^1)^2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix},$$

respectively. Thus, from

$$t^{(i)}_{(j)} = \sqrt{\frac{g_{ii}}{g_{jj}}}\, t^i_j,$$

we find

$$t^{(\underline{i})}_{(\underline{i})} = t^{\underline{i}}_{\underline{i}}, \qquad t^{(2)}_{(3)} = t^{(3)}_{(2)} = 0,$$

and

$$t^{(1)}_{(2)} = \sqrt{\frac{g_{11}}{g_{22}}} t^1_2 = \sqrt{\frac{9}{9(X^1)^2}} a(\underset{\sim}{X}) = \frac{a(\underset{\sim}{X})}{X^1},$$

$$t^{(1)}_{(3)} = \sqrt{\frac{g_{11}}{g_{33}}} t^1_3 = \sqrt{\frac{9}{4}} b(\underset{\sim}{X}) = \frac{3b(\underset{\sim}{X})}{2},$$

$$t^{(2)}_{(1)} = \sqrt{\frac{g_{22}}{g_{11}}} t^2_1 = \sqrt{\frac{9(X^1)^2}{9}} c(\underset{\sim}{X}) = X^1 c(\underset{\sim}{X}),$$

$$t^{(3)}_{(1)} = \sqrt{\frac{g_{33}}{g_{11}}} t^3_1 = \sqrt{\frac{4}{9}} d(\underset{\sim}{X}) = \frac{2d(\underset{\sim}{X})}{3},$$

Therefore, the physical tensor $t^{(i)}_{(j)}$ is given by

$$t^{(i)}_{(j)} = \begin{pmatrix} 1 & \frac{a(\underset{\sim}{X})}{X^1} & \frac{3b(\underset{\sim}{X})}{2} \\ X^1 c(\underset{\sim}{X}) & 1 & 0 \\ \frac{2d(\underset{\sim}{X})}{3} & 0 & e(\underset{\sim}{X}) \end{pmatrix}.$$

Thus, as each element of a physical tensor has the same dimensions, then

$$\left[\frac{a(\underset{\sim}{X})}{X^1}\right] = \left[\frac{3b(\underset{\sim}{X})}{2}\right] = [X^1 c(\underset{\sim}{X})] = \left[\frac{2d(\underset{\sim}{X})}{3}\right] = [e(\underset{\sim}{X})] = [1] = \text{constant},$$

so that

(a) $a(\underset{\sim}{X})$ has dimensions of $[X^1] =$ length,
(b) $b(\underset{\sim}{X})$ has dimensions of $[1] =$ constant,
(c) $c(\underset{\sim}{X})$ has dimensions of $\left[\frac{1}{X^1}\right] = \text{length}^{-1}$,
(d) $d(\underset{\sim}{X})$ has dimensions of $[1] =$ constant,
(e) $e(\underset{\sim}{X})$ has dimensions of $[1] =$ constant.

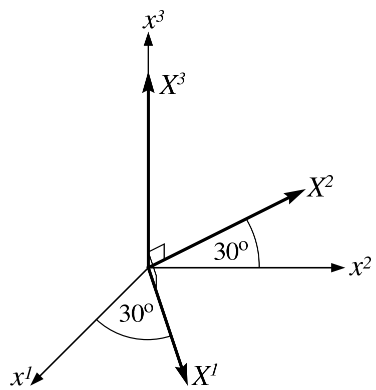
This in turn leads to the examples

(a) $a(\underset{\sim}{X}) = X^1 \cos X^2$,
(b) $b(\underset{\sim}{X}) = 42$,
(c) $c(\underset{\sim}{X}) = \dfrac{X^2}{X^1 + X^3}$,
(d) $d(\underset{\sim}{X}) = 0$,
(e) $e(\underset{\sim}{X}) = \mathrm{e}^{(X^1+X^3)^2/(X^1X^3)}$,

given that $[X^1] = [X^3] =$ length and $[X^2] =$ [angle] $=$ constant. Again, these examples are not unique.

**Question?**
Let $(x^1, x^2, x^3)$ denote the usual Cartesian coordinate system and $(X^1, X^2, X^3)$ denote an orthogonal curvilinear coordinate system, where both are centered at the same origin $O$.

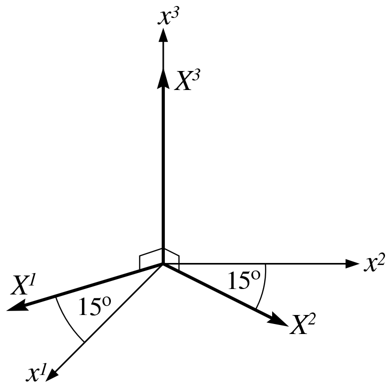
1.




In this case, the direction cosines $\alpha^i_j = \cos\angle_{X^i,x^j}$ are

$$\underset{\sim}{\alpha} = \begin{pmatrix} \cos 30^\circ & \cos 60^\circ & \cos 90^\circ \\ \cos 120^\circ & \cos 30^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

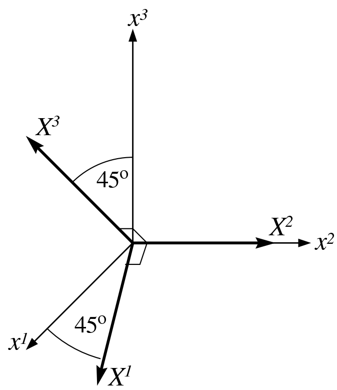
2.




In this case, the direction cosines $\alpha^i_j = \cos\angle_{X^i,x^j}$ are

$$\underset{\sim}{\alpha} = \begin{pmatrix} \cos 15^\circ & \cos 105^\circ & \cos 90^\circ \\ \cos 75^\circ & \cos 15^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2+\sqrt{3}}}{2} & -\frac{\sqrt{2-\sqrt{3}}}{2} & 0 \\ \frac{\sqrt{2-\sqrt{3}}}{2} & \frac{\sqrt{2+\sqrt{3}}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.




In this case, the direction cosines $\alpha^i_j = \cos\angle_{X^i,x^j}$ are

$$\underset{\sim}{\alpha} = \begin{pmatrix} \cos 45^\circ & \cos 90^\circ & \cos 135^\circ \\ \cos 90^\circ & \cos 0^\circ & \cos 90^\circ \\ \cos 45^\circ & \cos 90^\circ & \cos 45^\circ \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$