

MATH312 Tutorial Solutions
Autumn 2008
Week 4

Question 1:

1. Let $(z^1, z^2, z^3) = (x, y, z)$ and $(X^1, X^2, X^3) = (r, \theta, z)$. Thus,

$$\begin{aligned} g_{11} &= \frac{\partial z^k}{\partial X^1} \frac{\partial z^k}{\partial X^1} = \cos^2 \theta + \sin^2 \theta = 1, \\ g_{12} &= \frac{\partial z^k}{\partial X^1} \frac{\partial z^k}{\partial X^2} = -r \sin \theta \cos \theta + \sin \theta (r \cos \theta - 1) = -\sin \theta = g_{21}, \\ g_{13} &= \frac{\partial z^k}{\partial X^1} \frac{\partial z^k}{\partial X^3} = 0 = g_{31}, \\ g_{22} &= \frac{\partial z^k}{\partial X^2} \frac{\partial z^k}{\partial X^2} = r^2 \sin^2 \theta + (r \cos \theta - 1)^2 = r^2 \sin^2 \theta + r^2 \cos^2 \theta - 2r \cos \theta + 1 \\ &= r^2 - 2r \cos \theta + 1, \\ g_{23} &= \frac{\partial z^k}{\partial X^2} \frac{\partial z^k}{\partial X^3} = 0 = g_{32}, \\ g_{33} &= \frac{\partial z^k}{\partial X^3} \frac{\partial z^k}{\partial X^3} = 0 = 1. \end{aligned}$$

Therefore, the metric tensor is given by

$$g_{ij} = \begin{pmatrix} 1 & -\sin \theta & 0 \\ -\sin \theta & r^2 - 2r \cos \theta + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. We first need to find the determinant of the metric tensor, g . Thus

$$g = \begin{vmatrix} 1 & -\sin \theta & 0 \\ -\sin \theta & r^2 - 2r \cos \theta + 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = r^2 - 2r \cos \theta + 1 - \sin^2 \theta = (r - \cos \theta)^2.$$

Then,

$$\begin{aligned} g^{11} &= \frac{(-1)^{1+1}}{(r - \cos \theta)^2} \begin{vmatrix} r^2 - 2r \cos \theta + 1 & 0 \\ 0 & 1 \end{vmatrix} = \frac{r^2 - 2r \cos \theta + 1}{(r - \cos \theta)^2}, \\ g^{12} &= \frac{(-1)^{1+2}}{(r - \cos \theta)^2} \begin{vmatrix} -\sin \theta & 0 \\ 0 & 1 \end{vmatrix} = \frac{\sin \theta}{(r - \cos \theta)^2} = g^{21}, \\ g^{22} &= \frac{(-1)^{2+2}}{(r - \cos \theta)^2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{(r - \cos \theta)^2}, \\ g^{23} &= \frac{(-1)^{2+3}}{(r - \cos \theta)^2} \begin{vmatrix} 1 & -\sin \theta \\ 0 & 0 \end{vmatrix} = 0 = g_{32}, \\ g^{33} &= \frac{(-1)^{3+3}}{(r - \cos \theta)^2} \begin{vmatrix} 1 & -\sin \theta \\ -\sin \theta & r^2 - 2r \cos \theta + 1 \end{vmatrix} = 1. \end{aligned}$$

Thus, the conjugate metric tensor is given by

$$g^{jk} = \begin{pmatrix} \frac{r^2 - 2r \cos \theta + 1}{(r - \cos \theta)^2} & \frac{\sin \theta}{(r - \cos \theta)^2} & 0 \\ \frac{\sin \theta}{(r - \cos \theta)^2} & \frac{1}{(r - \cos \theta)^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. To find the Jacobian determinant J , we can use the relationship $J = \sqrt{g}$, so that

$$J = \sqrt{(r - \cos \theta)^2} = r - \cos \theta.$$

Question 2:

1. A_{ij} is a covariant tensor of order 2, which satisfies the transformation law

$$A_{mn}(X) = A_{ij}(x) \frac{\partial x^i}{\partial X^m} \frac{\partial x^j}{\partial X^n}.$$

2. B_j^i is a mixed tensor of order 2 with order one contravariant and order one covariant indices, which satisfies the transformation law

$$B_n^m(X) = B_j^i(x) \frac{\partial X^m}{\partial x^i} \frac{\partial x^j}{\partial X^n}.$$

3. C is a scalar tensor (of order 0), which satisfies the transformation law

$$C(X) = C(x).$$

4. ε_{ijk} is a covariant tensor of order 3, which satisfies the transformation law

$$\varepsilon_{mnp}(X) = \varepsilon_{ijk}(x) \frac{\partial x^i}{\partial X^m} \frac{\partial x^j}{\partial X^n} \frac{\partial x^k}{\partial X^p}.$$

5. I_{bye}^{hi} is a mixed tensor of order 5 with two contravariant and three covariant indices, which satisfies the transformation law

$$I_{rst}^{mn}(X) = I_{bye}^{hi}(x) \frac{\partial x^b}{\partial X^r} \frac{\partial x^y}{\partial X^s} \frac{\partial x^e}{\partial X^t} \frac{\partial X^m}{\partial x^h} \frac{\partial X^n}{\partial x^i}.$$

Question 3:

1. The position vector is given by

$$\begin{aligned} \underline{r} &= x \underline{e}_1 + y \underline{e}_2 + z \underline{e}_3 \\ &= a \sinh u \sin v \cos \phi \underline{e}_1 + a \sinh u \sin v \sin \phi \underline{e}_2 + a \cosh u \cos v \underline{e}_3. \end{aligned}$$

2. We know

$$\underline{E}_i = \frac{\partial \underline{r}}{\partial X^i},$$

so that if $(X^1, X^2, X^3) = (u, v, \phi)$, then

$$\underline{E}_1 = \frac{\partial \underline{r}}{\partial X^1} = a \cosh u \sin v \cos \phi \underline{e}_1 + a \cosh u \sin v \sin \phi \underline{e}_2 + a \sinh u \cos v \underline{e}_3.$$

$$\underline{E}_2 = \frac{\partial \underline{r}}{\partial X^2} = a \sinh u \cos v \cos \phi \underline{e}_1 + a \sinh u \cos v \sin \phi \underline{e}_2 - a \cosh u \sin v \underline{e}_3.$$

$$\underline{E}_3 = \frac{\partial \underline{r}}{\partial X^3} = -a \sinh u \sin v \sin \phi \underline{e}_1 + a \sinh u \sin v \cos \phi \underline{e}_2.$$

3. Using $g_{ij} = \underline{E}_i \cdot \underline{E}_j$, we find

$$\begin{aligned} g_{11} &= \underline{E}_1 \cdot \underline{E}_1 = a^2 \cosh^2 u \sin^2 v \cos^2 \phi + a^2 \cosh^2 u \sin^2 v \sin^2 \phi + a^2 \sinh^2 u \cos^2 v \\ &= a^2 (\cosh^2 u - \cos^2 v), \end{aligned}$$

$$\begin{aligned} g_{12} &= \underline{E}_1 \cdot \underline{E}_2 = a^2 \sinh u \cosh u \sin v \cos v \cos^2 \phi + a^2 \cosh u \sinh u \cos v \sin v \sin^2 \phi \\ &\quad - a^2 \sinh u \cosh u \sin v \cos v = 0 = g_{21}, \end{aligned}$$

$$\begin{aligned} g_{22} &= \underline{E}_2 \cdot \underline{E}_2 = a^2 \sinh^2 u \cos^2 v \cos^2 \phi + a^2 \sinh^2 u \cos^2 v \sin^2 \phi + a^2 \cosh^2 u \sin^2 v \\ &= a^2 (\cosh^2 u - \cos^2 v), \end{aligned}$$

$$\begin{aligned} g_{23} &= \underline{E}_2 \cdot \underline{E}_3 = -a^2 \sinh^2 u \sin v \cos v \sin \phi \cos \phi + a^2 \sinh^2 u \sin v \cos v \sin \phi \cos \phi + 0 \\ &= 0 = g_{32}, \end{aligned}$$

$$\begin{aligned} g_{33} &= \underline{E}_3 \cdot \underline{E}_3 = a^2 \sinh^2 u \sin^2 v \sin^2 \phi + \sinh^2 u \sin^2 v \cos^2 \phi \\ &= a^2 \sinh^2 u \sin^2 v. \end{aligned}$$

Therefore, we found

$$g_{ij} = \begin{pmatrix} a^2(\cosh^2 u - \cos^2 v) & 0 & 0 \\ 0 & a^2(\cosh^2 u - \cos^2 v) & 0 \\ 0 & 0 & a^2 \sinh^2 u \sin^2 v \end{pmatrix}.$$

Question 4:

1.

$$T^i = g^{ij} T_j.$$

2.

$$H_{bc}^a = g_{tb} g_{sc} g^{ia} H_i^{ts}.$$

3.

$$D_w^{rstuv} = g^{ar} g^{bs} g^{ct} g^{du} g^{ev} g_{jw} D_{abcde}^j.$$

4.

$$g_{mn} = g_{jm} g_{kn} g^{jk}.$$

5. Let $\Gamma_j^i = \frac{\partial x^i}{\partial X^j}$. Then,

$$\Gamma_n^m = g^{jm} g_{in} \Gamma_j^i$$

so that

$$\frac{\partial X^m}{\partial x^n} = g^{jm} g_{in} \frac{\partial x^i}{\partial X^j}.$$

Question 5:

Consider the cylindrical polar coordinates transformation

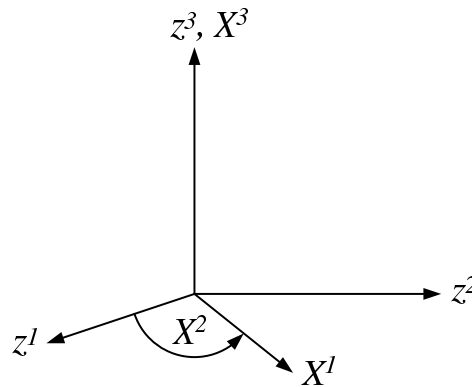
$$z^1 = X^1 \cos X^2, \quad z^2 = X^1 \sin X^2, \quad z^3 = X^3,$$

and let the vector \underline{A} be represented by

$$\underline{A} = A_x \underline{e}_1 + A_y \underline{e}_2 + A_z \underline{e}_3.$$

Then:

1.



2. The inverse transformation is given by considering:

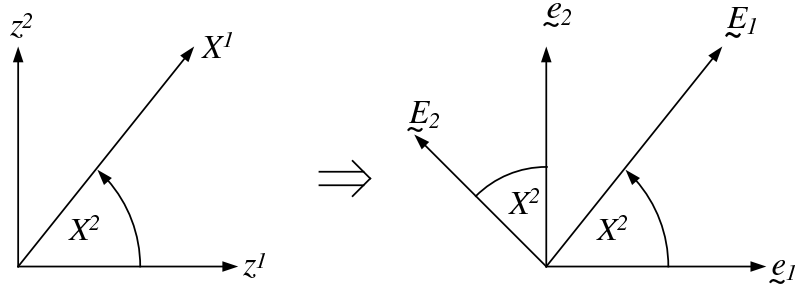
$$\begin{aligned} (z^1)^2 + (z^2)^2 &= (X^1)^2 \cos^2 X^2 + (X^1)^2 \sin^2 X^2 &\Rightarrow X^1 &= \sqrt{(z^1)^2 + (z^2)^2}. \\ \frac{z^2}{z^1} &= \frac{X^1 \sin X^2}{X^1 \cos X^2} &\Rightarrow X^2 &= \arctan \frac{z^2}{z^1}. \\ z^3 &= X^3 &\Rightarrow X^3 &= z^3. \end{aligned}$$

The inverse transformation exists when the Jacobian determinant J is non-zero, i.e.,

$$J = \left| \frac{\partial z^i}{\partial X^j} \right| = \begin{vmatrix} \cos X^2 & \sin X^2 & 0 \\ -X^1 \sin X^2 & X^1 \cos X^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = X^1 \neq 0.$$

Therefore, the inverse transformation exists provided $X^1 \neq 0$.

3. Consider the coordinate system, i.e.,



Now, if we “add up” the horizontal components, we find

$$e_1 = \tilde{E}_1 \cos X^2 - \tilde{E}_2 \sin X^2.$$

Similarly, from the vertical components, we find

$$e_2 = \tilde{E}_2 \sin X^2 + \tilde{E}_1 \cos X^2.$$

Alternatively, you could consider adding up the components along the X^1 and X^2 directions, which gives

$$\begin{aligned} \tilde{E}_1 &= e_1 \cos X^2 + e_2 \sin X^2, \\ \tilde{E}_2 &= -e_1 \sin X^2 + e_2 \cos X^2, \end{aligned}$$

respectively. And, clearly, $\tilde{E}_3 = e_3$.

4. To show that the tangential basis vectors are unit vectors, we need to show that $|\tilde{E}_i| = 1$. Thus,

$$\begin{aligned} |\tilde{E}_1| &= \sqrt{\cos^2 X^2 + \sin^2 X^2 + 0^2} = 1, \\ |\tilde{E}_2| &= \sqrt{\sin^2 X^2 + \cos^2 X^2 + 0^2} = 1, \\ |\tilde{E}_3| &= \sqrt{0^2 + 0^2 + 1} = 1. \end{aligned}$$

Therefore, the tangential basis vectors are unit vectors.

5. To find the contravariant physical components of \underline{A} , consider $A^{(i)} = A^i |\tilde{E}_i|$. Thus, as the tangential basis vectors are unit vectors, then the contravariant components of \underline{A} are precisely the physical components. To calculate the contravariant components, we need to express the orthonormal basis vectors e_i in terms of the tangential basis vectors. Thus, from above, we find

$$\begin{aligned} e_1 &= \tilde{E}_1 \cos X^2 - \tilde{E}_2 \sin X^2, \\ e_2 &= \tilde{E}_1 \sin X^2 + \tilde{E}_2 \cos X^2. \end{aligned}$$

Therefore, the vector \underline{A} can be expressed in terms of \tilde{E}_i as

$$\begin{aligned} \underline{A} &= A_x e_1 + A_y e_2 + A_z e_3, \\ &= A_x (\tilde{E}_1 \cos X^2 - \tilde{E}_2 \sin X^2) + A_y (\tilde{E}_1 \sin X^2 + \tilde{E}_2 \cos X^2) + A_z \tilde{E}_3, \\ &= (A_x \cos X^2 + A_y \sin X^2) \tilde{E}_1 - (A_x \sin X^2 - A_y \cos X^2) \tilde{E}_2 + A_z \tilde{E}_3. \end{aligned}$$

Hence, the physical components (and the contravariant components) of \underline{A} are

$$\begin{aligned} A^{(1)} &= A^1 = A_x \cos X^2 + A_y \sin X, \\ A^{(2)} &= A^2 = -A_x \sin X^2 + A_y \cos X^2, \\ A^{(3)} &= A^3 = A_z. \end{aligned}$$

6. To find the gradient basis vectors, we know

$$\underline{\tilde{E}}^1 = \frac{1}{V}(\underline{\tilde{E}}_2 \times \underline{\tilde{E}}_3), \quad \underline{\tilde{E}}^2 = \frac{1}{V}(\underline{\tilde{E}}_3 \times \underline{\tilde{E}}_1), \quad \underline{\tilde{E}}^3 = \frac{1}{V}(\underline{\tilde{E}}_1 \times \underline{\tilde{E}}_2),$$

where $V = \underline{\tilde{E}}_1 \cdot (\underline{\tilde{E}}_2 \times \underline{\tilde{E}}_3) \neq 0$ is the triple scalar product and represents the volume of the parallelepiped having the basis vectors for its sides. Note it is important that the order of indices remain in the same cycle (in a similar manner to the permutation symbol ε_{ijk}). If the order is not maintained, then V will be different. Now, as we know the tangential basis vectors $\underline{\tilde{E}}_i$ are unit vectors, then $V = 1$.

Hence, we now consider

$$\begin{aligned} \underline{\tilde{E}}^1 &= \underline{\tilde{E}}_2 \times \underline{\tilde{E}}_3 = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ -\sin X^2 & \cos X^2 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \\ &= \cos X^2 \underline{e}_1 + \sin X^2 \underline{e}_2, \\ &= \underline{\tilde{E}}_1. \end{aligned}$$

Similarly, noting the cyclic order of the indices, we consider

$$\begin{aligned} \underline{\tilde{E}}^2 &= \underline{\tilde{E}}_3 \times \underline{\tilde{E}}_1 = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ 0 & 0 & 1 \\ \cos X^2 & \sin X^2 & 0 \end{vmatrix}, \\ &= -\sin X^2 \underline{e}_1 + \cos X^2 \underline{e}_2, \\ &= \underline{\tilde{E}}_2. \end{aligned}$$

Finally, to find $\underline{\tilde{E}}^3$, we consider

$$\begin{aligned} \underline{\tilde{E}}^3 &= \underline{\tilde{E}}_1 \times \underline{\tilde{E}}_2 = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \cos X^2 & \sin X^2 & 0 \\ -\sin X^2 & \cos X^2 & 0 \end{vmatrix}, \\ &= \underline{e}_3, \\ &= \underline{\tilde{E}}_3. \end{aligned}$$

Therefore, the gradient basis vectors are respectively equal to their corresponding gradient basis vectors.

Note: You should check that the tangential and gradient basis vectors satisfy the reciprocal basis condition $\underline{\tilde{E}}_i \cdot \underline{\tilde{E}}^j = \delta_i^j$.

7. As the gradient basis vectors are respectively equal to their corresponding gradient basis vectors, then the covariant physical components are respectively equal to their corresponding contravariant physical components, i.e.,

$$\begin{aligned}A_{(1)} &= A^{(1)} = A_x \cos X^2 + A_y \sin X^2, \\A_{(2)} &= A^{(2)} = -A_x \sin X^2 + A_y \cos X^2, \\A_{(3)} &= A^{(3)} = A_z.\end{aligned}$$