MATH312: Applied Mathematical Modelling III
Autumn 2008
Solutions of Surprise Test for Week 3

Question 1:

1.

$$\begin{aligned} \delta_{ij}\,\delta_{ij} &= \delta_{1j}\,\delta_{1j} + \delta_{2j}\,\delta_{2j} + \delta_{3j}\,\delta_{3j} && \text{summing dummy index } i \\ &= \delta_{11}\,\delta_{11} + \delta_{22}\,\delta_{22} + \delta_{33}\,\delta_{33} && \text{other } \delta_{ij}\text{'s zero} \\ &= 1 + 1 + 1 = 3. \end{aligned}$$

2.

$$\begin{aligned} \varepsilon_{ijk}\,\varepsilon_{jki} &= \varepsilon_{1jk}\,\varepsilon_{jk1} + \varepsilon_{2jk}\,\varepsilon_{jk2} + \varepsilon_{3jk}\,\varepsilon_{jk3} && \text{summing dummy index } i \\ &= \varepsilon_{12k}\,\varepsilon_{2k1} + \varepsilon_{13k}\,\varepsilon_{3k1} + \varepsilon_{21k}\,\varepsilon_{1k2} \\ &\quad + \varepsilon_{23k}\,\varepsilon_{3k2} + \varepsilon_{31k}\,\varepsilon_{1k3} + \varepsilon_{32k}\,\varepsilon_{2k3} && \text{summing } j\text{, other } \varepsilon_{ijk}\text{'s zero} \\ &= \varepsilon_{123}\,\varepsilon_{231} + \varepsilon_{132}\,\varepsilon_{321} + \varepsilon_{213}\,\varepsilon_{132} \\ &\quad + \varepsilon_{231}\,\varepsilon_{312} + \varepsilon_{312}\,\varepsilon_{123} + \varepsilon_{321}\,\varepsilon_{213} && \text{summing } k\text{, other } \varepsilon_{ijk}\text{'s zero} \\ &= 1 \times 1 + (-1) \times (-1) + (-1) \times (-1) \\ &\quad + 1 \times 1 + 1 \times 1 + (-1) \times (-1) \\ &= 6. \end{aligned}$$

Alternatively, you could use the $\varepsilon - \delta$ identity.

3.

$$\begin{aligned} \delta_{ij}\,\varepsilon_{ijk} &= \delta_{ii}\,\varepsilon_{iik} && i = j\text{, otherwise } \delta_{ij} = 0 \\ &= 0 && \text{because } \varepsilon_{iik} = 0 \text{ due to repeated index} \end{aligned}$$

Question 2:

1.

$$\begin{aligned} \underset{\sim}{g} \cdot \underset{\sim}{h} &= (g_i \underset{\sim}{e}_i) \cdot (h_j \underset{\sim}{e}_j) \\ &= g_i h_i (\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j) \\ &= g_i h_j \delta_{ij} \\ &= g_i h_i. \end{aligned}$$

2.

$$\begin{aligned} \underset{\sim}{h} \times \underset{\sim}{g} \cdot \underset{\sim}{f} &= [(h_i \underset{\sim}{e}_i) \times (g_j \underset{\sim}{e}_j)] \cdot (f_k \underset{\sim}{e}_k) \\ &= (h_i g_j \varepsilon_{ijn} \underset{\sim}{e}_n) \cdot (f_k \underset{\sim}{e}_k) \\ &= \varepsilon_{ijn} h_i g_j f_k (\underset{\sim}{e}_n \cdot \underset{\sim}{e}_k) \\ &= \varepsilon_{ijn} h_i g_j f_k \delta_{nk} \\ &= \varepsilon_{ijk} h_i g_j f_k. \end{aligned}$$

3.

$$\underset{\sim}{g} \times \underset{\sim}{f} \cdot \underset{\sim}{h} = \underset{\sim}{h} \times \underset{\sim}{g} \cdot \underset{\sim}{f}$$
$$= \varepsilon_{ijk} h_i g_j f_k.$$

4.

$$\begin{aligned}
\underset{\sim}{\nabla} \cdot \underset{\sim}{f} \times \underset{\sim}{g} &= \left(\underset{\sim}{e}_m \frac{\partial}{\partial x^m} \right) \cdot [(f_i \underset{\sim}{e}_i) \times (g_j \underset{\sim}{e}_j)] \\
&= \left(\underset{\sim}{e}_m \frac{\partial}{\partial x^m} \right) \cdot [f_i g_j \varepsilon_{ijn} \underset{\sim}{e}_n] \\
&= \varepsilon_{ijn} \frac{\partial}{\partial x^m} \{ f_i g_j \} (\underset{\sim}{e}_m \cdot \underset{\sim}{e}_n) \\
&= \varepsilon_{ijn} [f_{i,m} g_j + f_i g_{j,m}] \delta_{mn} \\
&= \varepsilon_{ijn} [f_{i,n} g_j + f_i g_{j,n}].
\end{aligned}$$

Question 3:

1.

$$J = \left| \frac{\partial x^i}{\partial X^j} \right| = \begin{vmatrix} \cos X^2 & -X^1 \sin X^2 & 0 \\ \sin X^2 & X^1 \cos X^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = X^1 (\cos^2 X^2 + \sin^2 X^2) = X^1.$$

2.

$$\begin{aligned}
\frac{\partial \Phi}{\partial X^2} &= \frac{\partial \Phi}{\partial x^i} \frac{\partial x^i}{\partial X^2} = \frac{\partial \Phi}{\partial x^1} \frac{\partial x^1}{\partial X^2} + \frac{\partial \Phi}{\partial x^2} \frac{\partial x^2}{\partial X^2} + \frac{\partial \Phi}{\partial x^3} \frac{\partial x^3}{\partial X^2} \\
&= -X^2 \sin X^2 \frac{\partial \Phi}{\partial x^1} + X^1 \cos X^2 \frac{\partial \Phi}{\partial x^2} + 0 \\
&= -x^2 \frac{\partial \Phi}{\partial x^1} + x^1 \frac{\partial \Phi}{\partial x^2}.
\end{aligned}$$

3.

$$\begin{aligned}
g_{11} &= \frac{\partial z^k}{\partial X^1} \frac{\partial z^k}{\partial X^1} = \frac{\partial z^1}{\partial X^1} \frac{\partial z^1}{\partial X^1} + \frac{\partial z^2}{\partial X^1} \frac{\partial z^2}{\partial X^1} + \frac{\partial z^3}{\partial X^1} \frac{\partial z^3}{\partial X^1} \\
&= \cos^2 X^2 + \sin^2 X^2 + 0 \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
g_{22} &= \frac{\partial z^k}{\partial X^2} \frac{\partial z^k}{\partial X^2} = \frac{\partial z^1}{\partial X^2} \frac{\partial z^1}{\partial X^2} + \frac{\partial z^2}{\partial X^2} \frac{\partial z^2}{\partial X^2} + \frac{\partial z^3}{\partial X^2} \frac{\partial z^3}{\partial X^2} \\
&= (X^1)^2 \sin^2 X^2 + (X^1)^2 \cos^2 X^2 + 0 \\
&= (X^1)^2.
\end{aligned}$$

4.

$$\begin{aligned}
g &= J^2 \\
&= (X^1)^2.
\end{aligned}$$

5. If metric tensor is a diagonal matrix, then

$$\begin{aligned}|g_{ij}| &= g = g_{11}g_{22}g_{33}\\ &\Rightarrow (X^1)^2 = 1 \times (X^1)^2 \times g_{33}\\ &\Rightarrow g_{33} = 1.\end{aligned}$$

Question 4:

The metric tensor becomes

$$g_{ij} = \begin{pmatrix} 1 & \sin\theta & 0 \\ \sin\theta & 1 + 2r\cos\theta + r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

while the conjugate metric tensor becomes

$$g^{jk} = \begin{pmatrix} \frac{1+2r\cos\theta+r^2}{(r+\cos\theta)^2} & -\frac{\sin\theta}{(r+\cos\theta)^2} & 0 \\ -\frac{\sin\theta}{(r+\cos\theta)^2} & \frac{1}{(r+\cos\theta)^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Question 5:

These are some possible answers - there are an infinite number of possible correct answers.

1. B, $\bar{B} = B$.
2. B_k, $\bar{B}_k = B_j \dfrac{\partial x^j}{\partial X^k}$.
3. B^m, $\bar{B}^n = B^m \dfrac{\partial X^n}{\partial x^m}$.
4. ε_{ijk}, $\bar{\varepsilon}_{rst} = \varepsilon_{ijk} \dfrac{\partial x^i}{\partial X^r} \dfrac{\partial x^j}{\partial X^s} \dfrac{\partial x^k}{\partial X^t}$.
5. δ^i_j, $\bar{\delta}^m_n = \delta^i_j \dfrac{\partial X^m}{\partial x^i} \dfrac{\partial x^j}{\partial X^n}$.

Question 6:

1.

$$
\begin{aligned}
& & (\lambda+\mu)\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}) + \mu\nabla^2\underset{\sim}{u} &= 0 \\
&\Rightarrow & (\lambda+\mu)\left(\underset{\sim}{e}_m\frac{\partial}{\partial x^m}\right)\left[\left(\underset{\sim}{e}_i\frac{\partial}{\partial x^i}\right)\cdot(u_j\underset{\sim}{e}_j)\right] + \mu\frac{\partial^2}{\partial x^k\partial x^k}(u_n\underset{\sim}{e}_n) &= 0 \\
&\Rightarrow & (\lambda+\mu)\left(\underset{\sim}{e}_m\frac{\partial}{\partial x^m}\right)\left[u_{j,i}(\underset{\sim}{e}_i\cdot\underset{\sim}{e}_j)\right] + \mu u_{n,kk}\underset{\sim}{e}_n &= 0 \\
&\Rightarrow & (\lambda+\mu)\left(\underset{\sim}{e}_m\frac{\partial}{\partial x^m}\right)\left[u_{j,i}\delta_{ij}\right] + \mu u_{n,kk}\underset{\sim}{e}_n &= 0 \\
&\Rightarrow & (\lambda+\mu)\left(\underset{\sim}{e}_m\frac{\partial}{\partial x^m}\right)u_{i,i} + \mu u_{n,kk}\underset{\sim}{e}_n &= 0 \\
&\Rightarrow & (\lambda+\mu)\underset{\sim}{e}_m u_{i,im} + \mu u_{n,kk}\underset{\sim}{e}_n &= 0 \\
&\text{i.e.,} & (\lambda+\mu)\underset{\sim}{e}_j u_{i,ij} + \mu u_{j,ii}\underset{\sim}{e}_j &= 0
\end{aligned}
$$