MATH312 Tutorial Solutions
Autumn 2008
Week 2

Question 1:

1. Valid: $A^i + B^i = C^i$.
2. Valid: $A^i B_j = C^i_j$.
3. Valid: $A^i B^m_{jk} + D^{im}_j E_k = C^{im}_{jk}$.
4. Invalid: different order tensors.

Question 2:

1.
$$\begin{aligned} \sigma_{ii} &= \sigma_{11} + \sigma_{22} + \sigma_{33} && \text{summing dummy index } i \\ &= 3 + 0 + 0 \\ &= 3. \end{aligned}$$

2.
$$\begin{aligned} \sigma_{ij}\,\sigma_{ij} &= \sigma_{1j}\,\sigma_{1j} + \sigma_{2j}\,\sigma_{2j} + \sigma_{3j}\,\sigma_{3j} && \text{summing dummy index } i \\ &= \sigma_{11}\,\sigma_{11} + \sigma_{12}\,\sigma_{12} + \sigma_{13}\,\sigma_{13} && \text{summing dummy index } j \\ &\quad + \sigma_{21}\,\sigma_{21} + \sigma_{22}\,\sigma_{22} + \sigma_{23}\,\sigma_{23} \\ &\quad + \sigma_{31}\,\sigma_{31} + \sigma_{32}\,\sigma_{32} + \sigma_{33}\,\sigma_{33} \\ &= 3^2 + 1^2 + 1^2 + 1^2 + 0^2 + 2^2 + 1^2 + 2^2 + 0^2 \\ &= 21. \end{aligned}$$

3.
$$\begin{aligned} \sigma_{1j}\,\sigma_{j2} &= \sigma_{11}\,\sigma_{12} + \sigma_{12}\,\sigma_{22} + \sigma_{13}\,\sigma_{32} && \text{summing dummy index } j \\ &= 3 \times 1 + 1 \times 0 + 1 \times 2 \\ &= 5. \end{aligned}$$

Question 3:

1.
$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3. \qquad \text{summing dummy index } i$$

2.
$$\begin{aligned} \delta_{ij}\,\delta_{ij} &= \delta_{1j}\,\delta_{1j} + \delta_{2j}\,\delta_{2j} + \delta_{3j}\,\delta_{3j} && \text{summing dummy index } i \\ &= \delta_{11}\,\delta_{11} + \delta_{22}\,\delta_{22} + \delta_{33}\,\delta_{33} && \text{other } \delta_{ij}\text{'s zero} \\ &= 1 + 1 + 1 = 3. \end{aligned}$$

3.
$$\begin{aligned}
\varepsilon_{ijk}\,\varepsilon_{jki} &= \varepsilon_{1jk}\,\varepsilon_{jk1} + \varepsilon_{2jk}\,\varepsilon_{jk2} + \varepsilon_{3jk}\,\varepsilon_{jk3} && \text{summing dummy index } i \\
&= \varepsilon_{12k}\,\varepsilon_{2k1} + \varepsilon_{13k}\,\varepsilon_{3k1} + \varepsilon_{21k}\,\varepsilon_{1k2} \\
&\quad + \varepsilon_{23k}\,\varepsilon_{3k2} + \varepsilon_{31k}\,\varepsilon_{1k3} + \varepsilon_{32k}\,\varepsilon_{2k3} && \text{summing } j\text{, other } \varepsilon_{ijk}\text{'s zero} \\
&= \varepsilon_{123}\,\varepsilon_{231} + \varepsilon_{132}\,\varepsilon_{321} + \varepsilon_{213}\,\varepsilon_{132} \\
&\quad + \varepsilon_{231}\,\varepsilon_{312} + \varepsilon_{312}\,\varepsilon_{123} + \varepsilon_{321}\,\varepsilon_{213} && \text{summing } k\text{, other } \varepsilon_{ijk}\text{'s zero} \\
&= 1\times 1 + (-1)\times(-1) + (-1)\times(-1) \\
&\quad + 1\times 1 + 1\times 1 + (-1)\times(-1) \\
&= 6.
\end{aligned}$$

4.
$$\begin{aligned}
& \varepsilon_{ijk}\,A_jA_k = \varepsilon_{ikj}\,A_kA_j && j,k \text{ dummy indices, so name irrelevant} \\
\Rightarrow\quad & \varepsilon_{ijk}\,A_jA_k = -\varepsilon_{ijk}\,A_jA_k && \varepsilon_{ijk} = -\varepsilon_{ikj} \text{ and } A_jA_k = A_kA_j \\
\Rightarrow\quad & 2\,\varepsilon_{ijk}\,A_jA_k = 0 \\
\text{i.e.,}\quad & \varepsilon_{ijk}\,A_jA_k = 0.
\end{aligned}$$

5.
$$\begin{aligned}
\delta_{ij}\,\delta_{jk} &= \delta_{i1}\,\delta_{1k} + \delta_{i2}\,\delta_{2k} + \delta_{i3}\,\delta_{3k} && \text{summing dummy index } j \\
&= \begin{cases} 0, & \text{if } i \neq k \\ 1, & \text{if } i = k \end{cases} \\
&= \delta_{ik}.
\end{aligned}$$

6.
$$\begin{aligned}
\delta_{ij}\,\varepsilon_{ijk} &= \delta_{ii}\,\varepsilon_{iik} && i = j\text{, otherwise } \delta_{ij} = 0 \\
&= 0 && \text{because } \varepsilon_{iik} = 0 \text{ due to repeated index}
\end{aligned}$$

Question 4:

1. i = free index, k = dummy index, where range for both = 1, 2, 3.
2. If an index is repeated in the same term, then the repeated index is summed over its range.
3.
$$\begin{aligned}
& G\left(\frac{\partial^2 u_i}{\partial x_k \partial x_k} + \frac{1}{1-2\nu}\frac{\partial^2 u_k}{\partial x_k \partial x_i}\right) + X_i = \rho\frac{\partial^2 u_i}{\partial t^2} \\
\Rightarrow\quad & G\left(\frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2} + \frac{1}{1-2\nu}\left[\frac{\partial^2 u_1}{\partial x_1 \partial x_i} + \frac{\partial^2 u_2}{\partial x_2 \partial x_i} + \frac{\partial^2 u_3}{\partial x_3 \partial x_i}\right]\right) + X_i = \rho\frac{\partial^2 u_i}{\partial t^2} \\
\Rightarrow\quad & G\left(\underset{\sim}{\nabla}^2 u_i + \frac{1}{1-2\nu}\frac{\partial}{\partial x_i}\left[\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right]\right) + X_i = \rho\frac{\partial^2 u_i}{\partial t^2}
\end{aligned}$$

so that the single equation becomes

$$\begin{aligned}
G\left(\underset{\sim}{\nabla}^2 u_1 + \frac{1}{1-2\nu}\frac{\partial}{\partial x_1}\left[\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right]\right) + X_1 &= \rho\frac{\partial^2 u_1}{\partial t^2} \\
G\left(\underset{\sim}{\nabla}^2 u_2 + \frac{1}{1-2\nu}\frac{\partial}{\partial x_2}\left[\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right]\right) + X_2 &= \rho\frac{\partial^2 u_2}{\partial t^2} \\
G\left(\underset{\sim}{\nabla}^2 u_3 + \frac{1}{1-2\nu}\frac{\partial}{\partial x_3}\left[\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right]\right) + X_3 &= \rho\frac{\partial^2 u_3}{\partial t^2}.
\end{aligned}$$

Note: In Cartesian coordinates (x, y, z), we have

$$
\begin{aligned}
x_1 &= x, & x_2 &= y, & x_3 &= z \\
X_1 &= X, & X_2 &= Y, & X_3 &= Z \\
u_1 &= u, & u_2 &= v, & u_3 &= w
\end{aligned}
$$

where the equations become

$$
\begin{aligned}
G\left(\underset{\sim}{\nabla}^2 u + \frac{1}{1-2\nu}\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]\right) + X &= \rho\frac{\partial^2 u}{\partial t^2} \\
G\left(\underset{\sim}{\nabla}^2 v + \frac{1}{1-2\nu}\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]\right) + Y &= \rho\frac{\partial^2 v}{\partial t^2} \\
G\left(\underset{\sim}{\nabla}^2 w + \frac{1}{1-2\nu}\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]\right) + Z &= \rho\frac{\partial^2 w}{\partial t^2}.
\end{aligned}
$$

4. There is no right or wrong answer for Ronald's article title.

Question 5:

Let $x = x_1$, $y = x_2$ and $z = x_3$. Then the equations become

$$
\begin{aligned}
\epsilon_{11} &= \frac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right] = \frac{1}{E}\left[\sigma_{11} + \nu\sigma_{11} - \nu(\sigma_{11} + \sigma_{22} + \sigma_{33})\right], \\
\epsilon_{22} &= \frac{1}{E}\left[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right] = \frac{1}{E}\left[\sigma_{22} + \nu\sigma_{22} - \nu(\sigma_{11} + \sigma_{22} + \sigma_{33})\right], \\
\epsilon_{33} &= \frac{1}{E}\left[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right] = \frac{1}{E}\left[\sigma_{33} + \nu\sigma_{33} - \nu(\sigma_{11} + \sigma_{22} + \sigma_{33})\right], \\
\epsilon_{12} &= \frac{1+\nu}{E}\sigma_{12}, \\
\epsilon_{23} &= \frac{1+\nu}{E}\sigma_{23}, \\
\epsilon_{13} &= \frac{1+\nu}{E}\sigma_{13}.
\end{aligned}
$$

so that upon using the Kronecker delta, the equations can be written as the single equation

$$
\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\,\delta_{ij}\,\sigma_{kk}.
$$