Stress distributions in highly frictional granular heaps

Ngamta Thamwattana, Grant M. Cox* and James M. Hill

Abstract. The practice of storing granular materials in stock piles occurs throughout the world in many industrial situations. As a result, there is much interest in predicting the stress distribution within a stock pile. In 1981, it was suggested from experimental work that the peak force at the base does not occur directly beneath the vertex of the pile, but at some intermediate point resulting in a ring of maximum pressure. With this in mind, any analytical solution pertaining to this problem has the potential to provide useful insight into this phenomenon. Here, we propose to utilize some recently determined exact parametric solutions of the governing equations for the continuum mechanical theory of granular materials for two and three-dimensional stock piles. These solutions are valid provided $\sin\phi = 1$, where ϕ is the angle of internal friction, and we term such materials as "highly frictional". We note that there exists materials possessing angles of internal friction around 60 to 65 degrees, resulting in values of $\sin\phi$ equal to around 0.87 to 0.91. Further, the exact solutions presented here are potentially the leading terms in a perturbation solution for granular materials for which $1-\sin\phi$ is close to zero. The model assumes that the stock pile is composed of two regions, namely an inner rigid region and an outer yield region. The exact parametric solution is applied to the outer yield region, and the solution is extended continuously into the inner rigid region. The results presented here extend previous work of the authors to the case of highly frictional granular solids.

1 Introduction

The common practice of storing granular materials in stock piles occurs throughout the world in many industrial situations, ranging from fine powders used by chemical industries through to large irregularly shaped ores in mining industries. As a result,

*Corresponding author. Tel.: +61-2-4221-4304 ; fax: +61-2-4221-4845.
E-mail address: gcox@uow.edu.au (G. M. Cox).

a deeper and more thorough understanding of the behaviour of granular materials in a stock pile is essential for such industries. Further, the stress distribution throughout the stock pile, particularly the location of the maximum pressure at the base, enables the amount of settlement, caking, comminution and overall deterioration of the stored material to be predicted. Intuitively, we might expect the maximum pressure to lie directly beneath the vertex of the stock pile. However, experimental results predict otherwise (see, for example, Smid and Novosad [1]), where experiments instead indicate the existence of a ring of maximum pressure located at some intermediate point. This result has attracted much interest, including in the popular scientific literature (see, for example, Watson [2, 3]), and has produced numerous discrete and computational models that attempt to explain this curious phenomenon (see, for example, Bagster [4, 5, 6], Bagster and Li [7, 8], Bagster and Kirk [9], Brooks and Bagster [10], Liffman *et al.* [11] and Vanel *et al.* [12]). For further details, Savage [13] provides an extensive and critical review of the literature.

In this paper, we use the proper continuum mechanical theory of granular materials to predict the stress distribution in two-dimensional wedge and three-dimensional cone shaped stock piles. Following Hill and Cox [14], we propose a stock pile that is not entirely at yield, but is made up of an outer yield region, where the material is assumed to be at the limiting equilibrium point of yield, and an inner rigid region, where the material is assumed to be in equilibrium, but not at yield. Other multi-region stock piles have been considered elsewhere, for example, Cantelaube and Goddard [15], Cantelaube *et al.* [16] and Didwania *et al.* [17] consider "elasto-plastic" stock piles that contain an outer yield (plastic) region and an inner elastic region, and such stock piles do indeed predict a ring of maximum pressure. Numerical results of Hill and Cox [18] indicate that for a stock pile entirely at yield, the governing equations do not admit a solution, except for the special case of an angle of internal friction equal to ninety degrees. This may support the intuitive assumption that in general not all of the material in a stock pile can be simultaneously at yield. We note that the stock pile structure examined here has previously been considered in Hill and Cox [14]. However, in [14] it was tacitly assumed that $\phi \neq \pi/2$ and that the solution in the outer yield region followed the special simple exact solution first given by Sokolovsky [19], which has only one arbitrary constant of integration, and the stresses in the inner rigid region were assumed to be linear in x and y. Since this simple solution is not defined for $\phi = \pi/2$, here we utilize recently derived exact parametric solutions that contain two arbitrary constants of integration, as given in Hill and Cox [18] and Cox and Hill [20], for two-dimensional wedge and three-dimensional cone shaped stock piles respectively. We note that the two-dimensional solution was first derived in Hill and Cox [21] for the problem of determining the stress distribution within a two-dimensional wedge shaped hopper. We also note that a similar approach for multi-region two-dimensional parabolic and three-dimensional cubic shaped stock piles has been considered in Thamwattana and Hill [22], who utilize a further new exact parametric solution for an angle of internal friction equal to ninety degrees. This new solution is derived in Thamwattana and Hill [23] for two-dimensional parabolic and three-dimensional cubic shaped rat-holes.

The exact parametric solutions utilized here are only valid provided $\sin\phi = 1$, where ϕ is the angle of internal friction. There exists many granular materials that possess angles of internal friction around 60 to 65 degrees (see, for example, Australian Standard [24], Perkins [25, 26] and Sture [27]). These give rise to values of $\sin\phi$ equal to around 0.87 to 0.91, and we term such materials as "highly frictional". Further, the exact parametric solutions for $\sin\phi = 1$ can be utilized as the leading term in a perturbation scheme, due to the fact that the governing equations can be expressed in the forms given by (2.8) and (2.23), for two and three-dimensional stock piles respectively. From these equations, it is clear that approximate perturbation solutions involving powers of $1 - \sin\phi$, as given by (2.13) and (2.28), are possible.

For highly frictional granular materials, and in particular for materials that possess an angle of internal friction equal to ninety degrees, upon examining the Coulomb-Mohr yield condition, namely

$$|\tau| \leq c - \sigma \tan\phi, \tag{1.1}$$

we observe that although $\tan\phi$ tends to infinity as ϕ tends to $\pi/2$, we assume that along the yield surface the normal component of compressive traction σ, which is assumed to be positive in tension, tends to zero in such a manner that the product remains finite, and in particular is equal to the cohesion c. As a result, the tangential component of traction τ also tends to zero as ϕ tends to $\pi/2$. This assumption is supported upon examining a Mohr circle diagram when the angle of internal friction is equal to ninety degrees, from which it is clear that yield can only occur provided both the normal and tangential components of traction are zero, as shown in Figure 1. Further, this emphasizes that even with an angle of internal friction equal to ninety degrees, inter-particle slip may still occur (see, for example, the discussion given by Lynch and Mason [28, 29]). Also, from the Mohr circle diagram, it is clear that the minimum principal stress in magnitude must be zero when the material is yielding, and that the highly frictional yield condition only limits the state of yield in such a way that no tension is possible. Alternatively, from Spencer [30], we find that from the Mohr circle diagram we are able to deduce the expressions $\tau = q\cos\phi$ and $\sigma = q\sin\phi - p$, so that when $\phi = \pi/2$, the Coulomb-Mohr yield condition (2.6) becomes simply $p = q$, and τ and σ become zero, where $p = (\sigma_I + \sigma_{III})/2$, $q = (\sigma_I - \sigma_{III})/2$ and σ_I and σ_{III} denote the maximum and minimum principal stresses respectively. Further, we note that for materials possessing an angle of internal friction equal to ninety degrees, the behaviour of the material is completely dominated by the high value of the angle of internal friction, to the extent that cohesion becomes irrelevant, and indeed, in the following sections we see that formally in this limit the cohesion is eliminated from the governing equations. We also note that for an angle of internal friction equal to ninety degrees, the two families of generally distinct slip-planes coincide.

We further comment that as ϕ tends to $\pi/2$, then in terms of the Cauchy stresses the limiting yield condition (1.1) becomes $\sigma_{xy}^2 = \sigma_{xx}\sigma_{yy}$ and $\sigma_{rz}^2 = \sigma_{rr}\sigma_{zz}$, for two and three-dimensional stock piles respectively. We observe that these conditions also happen to be the natural conditions underlying all free surface problems, in the sense that the Cauchy stresses on any free surface must satisfy (3.31) and (3.52), which are only meaningful provided the expressions $\sigma_{xy}^2 - \sigma_{xx}\sigma_{yy}$ and $\sigma_{rz}^2 - \sigma_{rr}\sigma_{zz}$ vanish along the free surface, for two and three-dimensions respectively. As a result, the special case of an angle of internal friction equal to ninety degrees is reasonable as an initial approach for any problems that possesses a free surface, which indeed is the case for stock piles. If we assume these conditions are satisfied throughout the entire material, as is the case for $\phi = \pi/2$, then any valid solution has potential free surfaces at any point throughout the entire material.

In the following section, we state briefly the basic equations of the continuum mechanical theory of granular materials for steady quasi-static gravity flow according to the Coulomb-Mohr yield theory, for both two-dimensional plane strain and three-dimensional axially symmetric stock piles. In section 3, we apply the exact parametric solutions for the special case of an angle of internal friction equal to ninety degrees to the problem of determining the stress distribution in two and three-dimensional stock piles. Finally, the corresponding stress profiles and conclusion are presented in section 4.

2 Basic equations

In the following two subsections we briefly state the two and three-dimensional basic equations according to the continuum mechanical theory of granular materials for quasi-static steady gravity flow.

2.1 Two-dimensional basic equations

In the usual rectangular Cartesian coordinates (x, y) defined by Figure 2(b), assuming steady quasi-static plane strain flow enables the inertia terms to be neglected, so that the non-zero physical Cauchy stress components satisfy the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \rho g, \tag{2.2}$$

where ρ denotes the bulk density, g is acceleration due to gravity, both assumed constant, and σ_{xx}, σ_{xy} and σ_{yy} denote the usual physical Cauchy stress components, which are assumed to be positive in tension. These components can be decomposed in the standard form

$$\sigma_{xx} = -p + q\cos 2\psi, \quad \sigma_{yy} = -p - q\cos 2\psi, \quad \sigma_{xy} = q\sin 2\psi, \tag{2.3}$$

where p and q are the generally positive stress invariants defined by

$$p = -\frac{1}{2}(\sigma_{xx} + \sigma_{yy}), \quad q = \frac{1}{2}\{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2\}^{1/2}, \tag{2.4}$$

while ψ is the stress angle given by

$$\tan 2\psi = \frac{2\sigma_{xy}}{(\sigma_{xx} - \sigma_{yy})}, \tag{2.5}$$

where physically ψ is the angle between the direction of maximum principal stress and the x axis, in the direction of increasing φ, where $\varphi = \arctan(y/x)$. The decomposition of the stresses enables the equilibrium equations to be expressed in terms of just two unknowns, provided we complete the stress relations by assuming that the material is yielding according to the Coulomb-Mohr yield condition (1.1), which can be expressed in the form

$$q \leq p \sin\phi + c \cos\phi, \tag{2.6}$$

where c is the cohesion and ϕ denotes the angle of internal friction, both assumed constant. We note that equality only holds in (2.6) provided the granular material is at yield. The above equations are generally accepted as a reasonable basis for the determination of the stress distribution.

Now, upon substituting (2.3) and (2.6) into (2.2), and solving for q_x and q_y, we obtain

$$\begin{aligned} q_x &= \frac{\beta}{\beta^2 - 1}\left\{\rho g \beta \sin 2\psi + 2q\left[\psi_x \sin 2\psi - \psi_y\left(\beta + \cos 2\psi\right)\right]\right\}, \\ q_y &= \frac{\beta}{\beta^2 - 1}\left\{\rho g\left(1 - \beta\cos 2\psi\right) + 2q\left[\psi_x\left(\beta - \cos 2\psi\right) - \psi_y \sin 2\psi\right]\right\}, \end{aligned} \tag{2.7}$$

from which it is clear that $\beta = \pm 1$ gives rise to special cases, where $\beta = \sin\phi$. Upon rewriting (2.7) in the form

$$\begin{aligned} (\beta - 1)(q_x \cos\psi + q_y \sin\psi) &= \rho g \beta \sin\psi + 2\beta q(\psi_x \sin\psi - \psi_y \cos\psi), \\ (\beta + 1)(q_x \sin\psi - q_y \cos\psi) &= \rho g \beta \cos\psi - 2\beta q(\psi_x \cos\psi + \psi_y \sin\psi), \end{aligned} \tag{2.8}$$

then for the special case of $\beta = 1$, it follows from $(2.8)_1$ that q is given explicitly by

$$q = -\frac{\rho g}{2} \frac{1}{(\psi_x - \psi_y \cot \psi)}, \tag{2.9}$$

while $(2.8)_2$ becomes

$$2(q \sin \psi)_x = \rho g \cos \psi + 2(q \cos \psi)_y. \tag{2.10}$$

Thus, upon substituting (2.9) into (2.10), and simplifying, we obtain the nonlinear partial differential equation

$$h_{xx} - 2hh_{xy} + h^2 h_{yy} = 0, \tag{2.11}$$

where $h(x, y) = \cot \psi$. We observe that in the other special case of $\beta = -1$, we may deduce the same equation (2.11) in a similar manner, but where $h(x, y) = -\tan \psi$. However, as this case is non-physical, it will not be considered further.

As an aside, we note that equations (2.8) can be rewritten in the form

$$\begin{aligned} &\rho g \sin \psi + 2q\,(\psi_x \sin \psi - \psi_y \cos \psi) \\ &= (1 - \beta)\left[\rho g \sin \psi - q_x \cos \psi - q_y \sin \psi + 2q\,(\psi_x \sin \psi - \psi_y \cos \psi)\right], \\ &\rho g \cos \psi - 2(q \sin \psi)_x + 2(q \cos \psi)_y \\ &= (1 - \beta)\left[\rho g \cos \psi - q_x \sin \psi + q_y \cos \psi - 2q\,(\psi_x \cos \psi + \psi_y \sin \psi)\right], \end{aligned} \tag{2.12}$$

from which it is clear to see that these equations admit perturbation solutions of the form

$$\psi = \psi_0(x, y) + \varepsilon \psi_1(x, y) + O(\varepsilon^2), \quad q = q_0(x, y) + \varepsilon q_1(x, y) + O(\varepsilon^2), \tag{2.13}$$

where $\varepsilon = 1 - \beta$, with (2.13) satisfying (2.9) and (2.10) to leading order.

We also comment that as we require the top surface of the stock pile to be stress free, then if we consider any free surface within the material $f(x, y) =$ constant, we have upon differentiating with respect to x

$$f_x + f_y \frac{dy}{dx} = 0, \tag{2.14}$$

which gives

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{n_x}{n_y}, \tag{2.15}$$

noting that the unit normal $\mathbf{n} = (n_x, n_y)$ to the free surface has components given by

$$n_x = \frac{f_x}{(f_x^2 + f_y^2)^{1/2}}, \quad n_y = \frac{f_y}{(f_x^2 + f_y^2)^{1/2}}. \tag{2.16}$$

Thus, from (2.3), (2.6), (2.15) and (3.31) we may deduce for the special case of $\phi = \pi/2$ that $dy/dx = -\cot\psi$, which coincides with the equation for the slip-planes.

2.2 Three-dimensional basic equations

In the cylindrical polar coordinates (r, z) defined by Figure 3(b), assuming steady quasi-static axially symmetric flow enables the inertia terms to be neglected, so that the non-zero physical Cauchy stress components satisfy the equilibrium equations

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad \frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = \rho g, \tag{2.17}$$

where ρ denotes the bulk density, g is acceleration due to gravity, both assumed constant, and σ_{rr}, σ_{rz} and σ_{zz} denote the usual physical in-plane Cauchy stress components and $\sigma_{\theta\theta}$ is the hoop stress, where all the stress components are assumed to be positive in tension. The in-plane components can be decomposed in the standard form

$$\sigma_{rr} = -p + q\cos 2\psi, \quad \sigma_{zz} = -p - q\cos 2\psi, \quad \sigma_{rz} = q\sin 2\psi, \tag{2.18}$$

where p and q are the generally positive stress invariants defined by

$$p = -\frac{1}{2}(\sigma_{rr} + \sigma_{zz}), \quad q = \frac{1}{2}\{(\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2\}^{1/2}, \tag{2.19}$$

while ψ is the stress angle given by

$$\tan 2\psi = \frac{2\sigma_{rz}}{(\sigma_{rr} - \sigma_{zz})}, \tag{2.20}$$

where physically ψ is the angle between the direction of maximum principal stress and the r axis, in the direction of increasing ϑ, where $\vartheta = \arctan(z/r)$. The decomposition of the stresses enables the equilibrium equations to be expressed in terms of just two unknowns, provided we complete the stress relations by assuming that the material is yielding according to the Coulomb-Mohr yield condition (1.1), which can be expressed in the form of (2.6), and assume a relationship for the hoop stress in terms of p, q and ψ. Here, we follow Cox *et al.* [31] and assume a stress state corresponding to one of the Haar-von Karman regimes, in particular

$$\sigma_{\theta\theta} = -p + q, \tag{2.21}$$

which states in axially symmetric problems that the cases of highest interest is when the hoop stress is equal to either the maximum or minimum principal stress. Here, we have assumed the former. The above relations are generally accepted as a reasonable basis for the determination of the stress components.

Now, upon substituting (2.6), (2.18) and (2.21) into (2.17), we obtain

$$\begin{aligned} q_r &= \frac{\beta}{\beta^2 - 1}\left\{\rho g \beta \sin 2\psi + 2q\left[\psi_r \sin 2\psi - \psi_z\left(\beta + \cos 2\psi\right)\right] + \frac{1}{r} q(\beta - 1)(\cos 2\psi - 1)\right\}, \\ q_z &= \frac{\beta}{\beta^2 - 1}\left\{\rho g\left(1 - \beta \cos 2\psi\right) + 2q\left[\psi_r\left(\beta - \cos 2\psi\right) - \psi_z \sin 2\psi\right] + \frac{1}{r} q(\beta - 1) \sin 2\psi\right\}, \end{aligned} \tag{2.22}$$

from which it is clear that $\beta = \pm 1$ gives rise to special cases, where $\beta = \sin\phi$. Upon rewriting (2.22) in the form

$$\begin{aligned} (\beta - 1)(q_r \cos\psi + q_z \sin\psi) &= \rho g \beta \sin\psi + 2\beta q(\psi_r \sin\psi - \psi_z \cos\psi), \\ (\beta + 1)(q_r \sin\psi - q_z \cos\psi) &= \rho g \beta \cos\psi - 2\beta q(\psi_r \cos\psi + \psi_z \sin\psi + \frac{1}{r}\sin\psi), \end{aligned} \tag{2.23}$$

then for the special case of $\beta = 1$, it follows from $(2.23)_1$ that q is given explicitly by

$$q = -\frac{\rho g}{2}\frac{1}{(\psi_r - \psi_z \cot\psi)}, \tag{2.24}$$

while $(2.23)_2$ becomes

$$2(q \sin\psi)_r = \rho g \cos\psi + 2(q \cos\psi)_z - \frac{2q}{r}\sin\psi. \tag{2.25}$$

Thus, upon substituting (2.24) into (2.25) and simplifying, we obtain the nonlinear partial differential equation

$$h_{rr} - 2hh_{rz} + h^2 h_{zz} - \frac{1}{r}(h_r - hh_z) = 0, \tag{2.26}$$

where $h(r,z) = \cot\psi$. We observe that in the other special case of $\beta = -1$, we may deduce the same equation (2.26) in a similar manner but where $h(r,z) = -\tan\psi$. However, as this case is non-physical, it will not be considered further.

As an aside, we note that equations (2.23) can be rewritten in the form

$$
\begin{aligned}
&\rho g \sin\psi + 2q\left(\psi_r \sin\psi - \psi_z \cos\psi\right) \\
&= (1-\beta)\left[\rho g \sin\psi - q_r \cos\psi - q_z \sin\psi + 2q\left(\psi_r \sin\psi - \psi_z \cos\psi\right)\right], \\
&\rho g \cos\psi - 2(q\sin\psi)_r + 2(q\cos\psi)_z - \frac{2q}{r}\sin\psi \\
&= (1-\beta)\left[\rho g \cos\psi - q_r \sin\psi + q_z \cos\psi - 2q\left(\psi_r \cos\psi + \psi_z \sin\psi + \frac{1}{r}\sin\psi\right)\right],
\end{aligned} \tag{2.27}
$$

from which it is clear to see that these equations admit perturbation solutions of the form

$$
\psi = \psi_0(r,z) + \varepsilon\psi_1(r,z) + O(\varepsilon^2), \quad q = q_0(r,z) + \varepsilon q_1(r,z) + O(\varepsilon^2), \tag{2.28}
$$

where $\varepsilon = 1 - \beta$, with (2.28) satisfying (2.24) and (2.25) to leading order.

3 The stock pile problem

In this section, we consider the problem of the determination of the stress distribution within a stock pile. To do this, we set up the coordinate axes at the vertex of the stock pile with gravity acting vertically downwards, as shown in Figures 2 and 3. In this case, we obtain the two-dimensional plane strain and three-dimensional axially symmetric governing equations derived in the previous section for steady quasi-static flow. We note that although the governing equations are for steady quasi-static flow of granular materials, the equations are valid for the static stock pile problem assuming that the material is on the point of yield. For the special case of $\beta = 1$, the governing equations reduce to (2.11) and (2.26) respectively, where both have recently been shown to admit an exact parametric solution with two arbitrary constants of integration (Cox *et al.* [31]). Here, we utilize these solutions to determine the stress distribution within a stock pile. We assume that the stock pile is not entirely at yield, but instead is comprised of an outer yield region, where the material is assumed to be at the limiting equilibrium point of yield, and an inner rigid region, where the material is assumed to be in equilibrium, but not at yield. In the outer yield region we utilize the exact parametric solutions for $\beta = 1$, and then extend the solution across the boundary into the inner rigid region in such a manner that the stresses are continuous. We assume the strict inequality of the yield condition in the inner rigid region, but there is no unique procedure for determining the form of the stress distribution in this region. However, we believe the procedure presented here is a "natural" extension of the solution applying in the outer yield region, in the sense that the functional form of the solution in the outer region is extended to the inner region.

3.1 Two-dimensional wedge stock pile

As shown in Cox *et al.* [31], the nonlinear equation (2.11) admits the following exact parametric solution

$$y = -\frac{\left[2e^{-s/2}s^{-1/2} + I(s)\right]x}{C_2}, \quad \cot\psi = \frac{I(s)}{C_2},$$
$$q = \frac{\rho g x}{4C_2}\frac{\left[I^2(s) + C_2^2\right]}{s^{1/2}e^{-s/2}}, \quad I(s) = \int^s \omega^{-1/2}e^{-\omega/2}d\omega + C_1, \tag{3.29}$$

where C_1 and C_2 denote arbitrary constants of integration. Upon introducing the arbitrary constant C_3, such that

$$I(s) = \int_0^s \omega^{-1/2}e^{-\omega/2}d\omega + C_3 = \sqrt{2\pi}\,\mathrm{erf}(s/2)^{1/2} + C_3, \tag{3.30}$$

enables the integral $I(s)$ to be expressed in terms of the usual error function erf. Now, in order to utilize this exact parametric solution, we need to make some assumptions about the stock pile. Firstly, we assume that the profile of the top free surface of the outer yield region is given by $y = -a|x|$, and that the boundary surface between the outer yield region and the inner rigid region is defined by $y = -b|x|$, where a and b are positive constants such that $0 < a < b < \infty$, as shown in Figure 2. Next, we need to make some assumptions about the appropriate boundary conditions.

Firstly, we consider the outer yield region. On the top surface of the stock pile, we assume a stress free surface. In this case, we require both the horizontal and vertical stresses to vanish, namely

$$\sigma_x = \sigma_{xx} n_x + \sigma_{xy} n_y = 0, \quad \sigma_y = \sigma_{xy} n_x + \sigma_{yy} n_y = 0, \tag{3.31}$$

where n_x and n_y denote the components of the normal to the top surface of the stock pile in the x and y directions respectively. On assuming a symmetrical stock pile, we need only consider one half of the stock pile, say $x \geq 0$. Thus, if the surface of the stock pile is defined by $y = -ax$, then we may deduce

$$n_x = \sin\lambda, \qquad n_y = \cos\lambda, \tag{3.32}$$

where λ is the angle between the top surface of the stock pile and the horizontal x axis, as defined in Figure 2(b), and as such $\tan\lambda = a$. Upon substituting (2.3), (2.6) and (3.32) into (3.31), recalling that $\phi = \pi/2$, then we obtain the boundary condition $\psi = -\lambda \pm \pi/2$ along the top surface of the stock pile $y = -ax$, which can be simplified to yield

$$\cot\psi = a, \quad \text{along} \quad y = -ax. \tag{3.33}$$

Now, upon applying the boundary condition (3.33) to $(3.29)_2$, we get

$$aC_2 = I(s_1), \tag{3.34}$$

where s_1 denotes the parameter value along the stock pile surface. Similarly, $(3.29)_1$ becomes

$$aC_2 = 2e^{-s_1/2} s_1^{-1/2} + I(s_1), \tag{3.35}$$

so that from (3.34) and (3.35) we must deduce $s_1 = \infty$. In this case, from (3.30) and either (3.34) or (3.35), we find that the arbitrary constant C_2 is given by

$$C_2 = \frac{\sqrt{2\pi} + C_3}{a}. \tag{3.36}$$

Thus, from (2.3), (2.6), (3.29), (3.30) and (3.36) we obtain expressions for the stresses within the outer region, namely

$$\sigma_{xx} = -\frac{\rho g[\sqrt{2\pi} + C_3]x}{2as^{1/2}e^{-s/2}}, \quad \sigma_{xy} = \frac{\rho g[\sqrt{2\pi}\mathrm{erf}(s/2)^{1/2} + C_3]x}{2s^{1/2}e^{-s/2}},$$
$$\sigma_{yy} = -\frac{\rho g a[\sqrt{2\pi}\mathrm{erf}(s/2)^{1/2} + C_3]^2 x}{2s^{1/2}e^{-s/2}[\sqrt{2\pi} + C_3]}, \tag{3.37}$$

where the only unknown constants are C_3 and b. Note that we assume the profile of the top surface is known, or in other words, a is assumed to be known. To check that the stresses given by (3.37) satisfy (3.31), we note that along any curve $y = -\chi x$ where $\chi \in [a, b]$, the normal to the surface according to (2.16) is given by $\mathbf{n} = (\ \chi/\sqrt{1+\chi^2}, 1/\sqrt{1+\chi^2}\ )$, so that from (3.31) and (3.37), noting $(3.29)_1$, we get

$$\sigma_x = -\frac{\rho g x}{s\sqrt{1+\chi^2}}, \quad \sigma_y = \frac{\rho g a[\sqrt{2\pi}\mathrm{erf}(s/2)^{1/2} + C_3]x}{s[\sqrt{2\pi} + C_3]\sqrt{1+\chi^2}}, \tag{3.38}$$

from which it is clear to see that the stress free condition (3.31) is satisfied when $s = s_1 = \infty$. Note that while we consider an infinite stock pile in height, we assume that at a finite height, say $y = -h$, the horizontal and vertical components of the stress vector will approximately be equal to a stock pile of height h resting on a horizontal rigid plane. With this in mind, we find that the normal to the surface $y = -h$ is given by $\mathbf{n} = (0, 1)$, so that the horizontal and vertical components of the stress vector are simply $\sigma_x = \sigma_{xy}$ and $\sigma_y = \sigma_{yy}$ respectively.

Next, we consider the inner rigid region, and assume that the stresses satisfy the equilibrium equations, but not the equality of the Coulomb-Mohr yield condition (2.6). Instead, we assume that the stresses in the inner rigid region satisfy the strict inequality of the yield condition, which from the definition of p and q given by (2.4), can be simplified to yield

$$\sigma_{xy}^2 < \sigma_{xx}\sigma_{yy}. \tag{3.39}$$

Now, upon examining the form of the stresses in the outer yield region given by (3.37), the natural extension of these stresses into the inner rigid region is to assume that they extend continuously into the inner rigid region, and this is easiest done by ensuring that the form of the stresses at the boundary between the two regions, namely $y = -bx$, are the same. Thus, after also ensuring that the equilibrium equations (2.2) are satisfied, we find

$$\sigma_{xx} = \rho g A y, \quad \sigma_{xy} = \rho g B x, \quad \sigma_{yy} = \rho g (1-B) y, \tag{3.40}$$

where A and B are constants to be determined. Thus, if we assume the parameter value $s = s_2$ corresponds to the boundary between the two regions, namely along $y = -bx$, then from (3.37) and (3.40), ensuring continuity requires

$$bA = \frac{[\sqrt{2\pi} + C_3]}{2as_2^{1/2}e^{-s_2/2}}, \qquad B = \frac{[\sqrt{2\pi}\mathrm{erf}(s_2/2)^{1/2} + C_3]}{2s_2^{1/2}e^{-s_2/2}},$$
$$b(1-B) = \frac{a[\sqrt{2\pi}\mathrm{erf}(s_2/2)^{1/2} + C_3]^2}{2s_2^{1/2}e^{-s_2/2}[\sqrt{2\pi} + C_3]}. \tag{3.41}$$

In this case, from $(3.41)_2$ we find

$$C_3 = 2Bs_2^{1/2}e^{-s_2/2} - \sqrt{2\pi}\mathrm{erf}(s_2/2)^{1/2}, \tag{3.42}$$

so that $(3.41)_1$ becomes

$$A = \frac{[\sqrt{2\pi}\mathrm{erfc}(s_2/2)^{1/2} + 2Bs_2^{1/2}e^{-s_2/2}]}{2abs_2^{1/2}e^{-s_2/2}}, \tag{3.43}$$

where erfc is the usual complementary error function. Finally, from $(3.41)_3$ and (3.42), we get

$$b = \frac{2aB^2 s_2^{1/2}e^{-s_2/2}}{(1-B)\left\{\sqrt{2\pi}\mathrm{erfc}(s_2/2)^{1/2} + 2Bs_2^{1/2}e^{-s_2/2}\right\}}. \tag{3.44}$$

Now, at the boundary between the two regions, namely $y = -bx$, we find that $(3.29)_1$ gives

$$bC_2 = 2s_2^{-1/2}e^{-s_2/2} + I(s_2), \tag{3.45}$$

so that from (3.44) and (3.45) we may deduce

$$s_2 = -\frac{(B-1)}{B(2B-1)}. \tag{3.46}$$

Thus, we are able to describe the stresses in the inner rigid region as given by (3.40), where A is given by (3.43), b is given by (3.44), s_2 is given by (3.46) and B is an arbitrary constant whose only constraint is that B must lie in either the range of $1/2 < B < 1$ or $B < 0$. However from numerical investigations, if $B < 0$, then from (3.44) we find that $b < 0$, which contradicts the assumption of $0 < a < b < \infty$. Finally, we note that as a result of ensuring the stresses are continuous across the boundary between the two regions, the arbitrary constant of integration C_3 is found to be given by (3.42).

Finally, we need to check that the stress solution in the inner rigid region satisfies the strict inequality of the yield condition, namely (3.39). Thus, from (3.40), (3.43) and (3.44), we find that (3.39) becomes

$$\left(\frac{y}{x}\right)^2 - b^2 > 0, \tag{3.47}$$

where we have used the relation $b^2 A(1-B) = B^2$. Clearly, (3.47) gives rise to the two possible conditions $y/x > b$ or $y/x < -b$, where the latter condition is satisfied throughout the entire inner rigid region, due to the inner rigid region being bounded by the surface $y/x = -b$, as shown in Figure 2. We note that we only consider $x \geq 0$ because we have assumed a symmetrical stock pile.

3.2 Three-dimensional conical stock pile

As shown in Cox *et al.* [31], the nonlinear equation (2.26) admits the following exact parametric solution

$$\begin{gathered} z = -\frac{\left[3s^{-1/3}e^{-s/3} + I(s)\right] r}{C_2}, \qquad \cot\psi = \frac{I(s)}{C_2}, \\ q = \frac{\rho g r}{6C_2} \frac{\left[I^2(s) + C_2^2\right]}{s^{2/3}e^{-s/3}}, \qquad I(s) = \int^s \omega^{-1/3} e^{-\omega/3} d\omega + C_1, \end{gathered} \tag{3.48}$$

where C_1 and C_2 denote arbitrary constants of integration. Upon introducing the arbitrary constant C_3, such that

$$I(s) = \int_0^s \omega^{-1/3} e^{-\omega/3} d\omega + C_3, \tag{3.49}$$

enables the integral $I(s)$ to be expressed in the form

$$I(s) = J(s) + C_3, \tag{3.50}$$

where $J(s)$ is the integral defined by

$$J(s) = \int_0^s \omega^{-1/3} e^{-\omega/3} d\omega, \tag{3.51}$$

so that $J(\infty) = 3^{2/3}\Gamma(2/3)$, where Γ is the usual gamma function. Now, in order to utilize this exact parametric solution, we need to make some assumptions about the stock pile. Firstly, we assume that the profile of the top surface of the outer yield region is given by $z = -a|r|$, and that the boundary surface between the outer yield region and the inner rigid region is defined by $z = -b|r|$, where a and b are positive constants such that $0 < a < b < \infty$, as shown in Figure 3. Next, we need to make some assumptions about the appropriate boundary conditions.

Firstly, we consider the outer yield region. On the top surface of the stock pile, we assume a stress free surface. In this case, we require both the horizontal and vertical stresses to vanish, namely

$$\sigma_r = \sigma_{rr} n_r + \sigma_{rz} n_z = 0, \quad \sigma_z = \sigma_{rz} n_r + \sigma_{zz} n_z = 0, \tag{3.52}$$

where n_r and n_z denote the components of the normal to the top surface of the stock pile in the r and z directions respectively. On assuming an axially symmetric stock pile, we need only consider one half slice of the stock pile, say $r \geq 0$ and $\theta = 0$. Thus, if the surface of the stock pile is defined by $z = -ar$, then we may deduce

$$n_r = \sin\lambda, \quad n_z = \cos\lambda, \tag{3.53}$$

where λ is the angle between the top surface of the stock pile and the horizontal r axis, as defined in Figure 3(b), and as such $\tan\lambda = a$. Upon substituting (2.6), (2.18) and (3.53) into (3.52), recalling that $\phi = \pi/2$, then we may obtain the boundary condition $\psi = -\lambda \pm \pi/2$ along the top surface of the stock pile $z = -ar$, which can be simplified to yield

$$\cot\psi = a, \quad \text{along} \quad z = -ar. \tag{3.54}$$

Now, upon applying the boundary condition (3.54) to $(3.48)_2$, we get

$$aC_2 = I(s_1), \tag{3.55}$$

where s_1 denotes the parameter value along the stock pile surface. Similarly, $(3.48)_1$ becomes

$$aC_2 = 3e^{-s_1/3}s_1^{-1/3} + I(s_1), \tag{3.56}$$

so that from (3.55) and (3.56) we must deduce $s_1 = \infty$. In this case, from (3.50) and either (3.55) or (3.56), we find that the arbitrary constant C_2 is given by

$$C_2 = \frac{3^{2/3}\Gamma(2/3) + C_3}{a}. \tag{3.57}$$

Thus, from (2.6), (2.18), (3.48), (3.50) and (3.57) we obtain expressions for the stresses within the outer yield region, namely

$$\sigma_{rr} = -\frac{\rho g[3^{2/3}\Gamma(2/3) + C_3]r}{3as^{2/3}e^{-s/3}}, \quad \sigma_{rz} = \frac{\rho g[J(s) + C_3]r}{3s^{2/3}e^{-s/3}},$$
$$\sigma_{zz} = -\frac{\rho g a[J(s) + C_3]^2 r}{3s^{2/3}e^{-s/3}[3^{2/3}\Gamma(2/3) + C3]}, \tag{3.58}$$

where the only unknown constants are C_3 and b. Note that we assume the profile of the top surface is known, or in other words, a is assumed to be known. To check that the stresses given by (3.58) satisfy (3.52), we note that along any curve $z = -\chi r$ where $\chi \in [a, b]$, the normal to the surface according to (2.16) is given by $\mathbf{n} = (\ \chi/\sqrt{1+\chi^2}, 1/\sqrt{1+\chi^2}\ )$, so that from (3.52) and (3.58), noting $(3.48)_1$, we get

$$\sigma_r = -\frac{\rho g r}{s\sqrt{1+\chi^2}}, \quad \sigma_z = \frac{\rho g a[J(s) + C_3]r}{s[3^{2/3}\Gamma(2/3) + C_3]\sqrt{1+\chi^2}}, \tag{3.59}$$

from which it is clear to see that the stress free condition (3.52) is satisfied when $s = s_1 = \infty$. We again note, that while we consider an infinite stock pile in height, we assume that at a finite height, say $z = -h$, the horizontal and vertical components of the stress vector will approximately be equal to a stock pile of height h resting on a horizontal rigid plane. With this in mind, we find that the normal to the surface $z = -h$ is given by $\mathbf{n} = (0, 1)$, so that the horizontal and vertical components of the stress vector are simply $\sigma_r = \sigma_{rz}$ and $\sigma_z = \sigma_{zz}$ respectively.

Next, we consider the inner rigid region, and assume that the stresses satisfy the equilibrium equations, but not the equality of the Coulomb-Mohr yield condition

(2.6). Instead, we assume that the stresses in the inner rigid region satisfy the strict inequality of the yield condition, which from the definition of p and q given by (2.19), can be simplified to yield

$$\sigma_{rz}^2 < \sigma_{rr}\sigma_{zz}. \tag{3.60}$$

Now, upon examining the form of the stresses in the outer yield region given by (3.58), the natural extension of these stresses into the inner rigid region is to assume that they extend continuously into the inner rigid region, and this is easiest done by ensuring that the form of the stresses at the boundary between the two regions, namely along $z = -br$, are the same. However, unlike the case for two-dimensional plane strain wedge stock pile, upon ensuring that the equilibrium equations (2.17) are satisfied for the three-dimensional axially symmetric stock pile, we obtain two possible forms for the stresses in the inner rigid region, namely the linear form

$$\sigma_{rr} = \rho g(Ar + Bz),\ \sigma_{rz} = \rho g Cr,\ \sigma_{zz} = \rho g(1 - 2C)z,\ \sigma_{\theta\theta} = \rho g(2Ar + Bz), \tag{3.61}$$

and the singular form

$$\sigma_{rr} = \rho g D\frac{z^2}{r}, \quad \sigma_{rz} = \rho g Er, \quad \sigma_{zz} = \rho g(1 - 2E)z, \quad \sigma_{\theta\theta} = 0, \tag{3.62}$$

where A, B, C, D and E are various constants to be determined. The two possible forms of the stresses arise because we assume that only the strict inequality of the yield condition (3.60) is satisfied, which again highlights the fact that there is no unique method of extending the stresses from the outer yield region continuously into the inner rigid region. The main difference between these two possible forms is the vanishing or otherwise of the hoop stress. Assuming that the hoop stress is non-zero in the inner rigid region, noting that in the outer yield region the hoop stress is zero, then on satisfying the equilibrium equations (2.17) a form of the stresses can be determined which is linear in both r and z, similar to the inner dead region of Hill and Cox [14]. However, assuming that the hoop stress is zero throughout the entire stock pile, then a form of the stresses arises which possesses a singularity in the Cauchy stress σ_{rr}. We comment that this is a weak singularity, in the sense that although σ_{rr} is singular, the integrated stress vector on any vanishingly small cylinder of radius r_0, remains finite as r_0 tends to zero. We note a similar weak singularity in σ_{rr} occurs for the three-dimensional axially symmetric cubic stock pile examined in Thamwattana and Hill [22]. In [22], the exact solution in the outer yield region possesses the singular behaviour in the σ_{rr} component, and this feature extends into the inner rigid region. Here, however, the exact solution in the outer yield region does not possess any singular behaviour in the σ_{rr} component, but it instead arises from the equilibrium equations (2.17) and assuming that the hoop stress remains zero throughout the entire stock pile. Although, the physical reasons underlying the existence of such a weak singularity are not properly understood, we comment that σ_{rr} does not contribute directly to the evaluation of the force distribution across the base of the stock pile.

3.2.1 Linear form given by (3.61)

Upon assuming that the stresses in the inner rigid region are of the linear form given by (3.61), and if we assume the parameter value $s = s_2$ corresponds to the boundary between the two regions, namely along $z = -br$, then from (3.58) and (3.61), ensuring continuity requires

$$A - bB = -\frac{[3^{2/3}\Gamma(2/3) + C_3]}{3as_2^{2/3}e^{-s_2/3}}, \quad C = \frac{[J(s_2) + C_3]}{3s_2^{2/3}e^{-s_2/3}},$$
$$b(1-2C) = \frac{a[J(s_2) + C_3]^2}{3s_2^{2/3}e^{-s_2/3}[3^{2/3}\Gamma(2/3) + C_3]}, \quad 2A - bB = 0. \tag{3.63}$$

In this case, from $(3.63)_2$ we find

$$C_3 = 3Cs_2^{2/3}e^{-s_2/3} - J(s_2), \tag{3.64}$$

so that from $(3.63)_4$, namely

$$B = \frac{2A}{b}, \tag{3.65}$$

we find that $(3.63)_1$ becomes

$$A = \frac{\left[3^{2/3}\Gamma(2/3) - J(s_2) + 3Cs_2^{2/3}e^{-s_2/3}\right]}{3as_2^{2/3}e^{-s_2/3}}. \tag{3.66}$$

Finally, from $(3.63)_3$ and (3.64), we get

$$b = \frac{3aC^2 s_2^{2/3} e^{-s_2/3}}{(1-2C)\left\{3^{2/3}\Gamma(2/3) - J(s_2) + 3Cs_2^{2/3}e^{-s_2/3}\right\}}, \tag{3.67}$$

so that (3.65) becomes

$$B = 2(1-2C)\left[\frac{3^{2/3}\Gamma(2/3) - J(s_2) + 3Cs_2^{2/3}e^{-s_2/3}}{3aCs_2^{2/3}e^{-s_2/3}}\right]^2. \tag{3.68}$$

Now, at the boundary between the two regions, namely $z = -br$, we find that $(3.48)_1$ gives

$$bC_2 = 3s_2^{-1/3}e^{-s_2/3} + I(s_2), \tag{3.69}$$

so that from (3.67) and (3.69) we may deduce

$$s_2 = -\frac{(2C-1)}{C(3C-1)}. \tag{3.70}$$

Thus, we are able to describe the stresses in the inner rigid region as given by (3.61), where A is given by (3.66), b is given by (3.67), B is given by (3.68), s_2 is given by (3.70) and C is an arbitrary constant whose only constraint is that C must lie in either the range of $1/3 < C < 1/2$ or $C < 0$. However from numerical investigations, if $C < 0$, then from (3.67) we find that $b < 0$, which contradicts the assumption of $0 < a < b < \infty$. Finally, we note that as a result of ensuring the stresses are continuous across the boundary between the two regions, the arbitrary constant C_3 is found to be given by (3.64).

Finally, we need to check that the stress solution in the inner rigid region satisfies the strict inequality of the yield condition, namely (3.60). Thus, from (3.61), (3.66), (3.67) and (3.68), we find that (3.60) becomes

$$2\left(\frac{z}{r}\right)^2 + b\left(\frac{z}{r}\right) - b^2 > 0, \tag{3.71}$$

where we have used the relation $bA(1-2C) = C^2$. Clearly, (3.71) gives rise to the two possible conditions $z/r > b/2$ or $z/r < -b$, where the latter condition is satisfied throughout the entire inner rigid region, due to the inner rigid region being bounded by the surface $z/r = -b$, as shown in Figure 3. We note that we only consider $r \geq 0$ because we have assumed a symmetrical stock pile.

3.2.2 Singular form given by (3.62)

Upon assuming that the stresses in the inner rigid region are of the singular form given by (3.62), and if we assume the parameter value $s = s_2$ corresponds to the boundary between the two regions, namely along $z = -br$, then from (3.58) and (3.62), ensuring continuity requires

$$b^2 D = -\frac{[3^{2/3}\Gamma(2/3) + C_3]}{3as_2^{2/3}e^{-s_2/3}}, \quad E = \frac{[J(s_2) + C_3]}{3s_2^{2/3}e^{-s_2/3}},$$
$$b(1-2E) = \frac{a[J(s_2) + C_3]^2}{3s_2^{2/3}e^{-s_2/3}[3^{2/3}\Gamma(2/3) + C_3]}. \tag{3.72}$$

In this case, from $(3.72)_2$ we find

$$C_3 = 3Es_2^{2/3}e^{-s_2/3} - J(s_2), \tag{3.73}$$

so that $(3.72)_1$ becomes

$$D = -\frac{[3^{2/3}\Gamma(2/3) - J(s_2) + 3Es_2^{2/3}e^{-s_2/3}]}{3ab^2 s_2^{2/3}e^{-s_2/3}}. \tag{3.74}$$

Finally, from $(3.72)_3$ and (3.73), we get

$$b = \frac{3aE^2 s_2^{2/3}e^{-s_2/3}}{(1-2E)\left\{3^{2/3}\Gamma(2/3) - J(s_2) + 3Es_2^{2/3}e^{-s_2/3}\right\}}. \tag{3.75}$$

Now, at the boundary between the two regions, namely $z = -br$, we find that $(3.48)_1$ gives

$$bC_2 = 3s_2^{-1/3}e^{-s_2/3} + I(s_2), \tag{3.76}$$

so that from (3.75) and (3.76) we may deduce

$$s_2 = -\frac{(2E-1)}{E(3E-1)}. \tag{3.77}$$

Thus, we are able to describe the stresses in the inner rigid region as given by (3.62), where D is given by (3.74), b is given by (3.75), s_2 is given by (3.77) and E is an arbitrary constant whose only constraint is that E must lie in either the range of $1/3 < E < 1/2$ or $E < 0$. However from numerical investigations, if $E < 0$, then from (3.75) we find that $b < 0$, which contradicts the assumption of $0 < a < b < \infty$. We note that as a result of ensuring the stresses are continuous across the boundary between the two regions, the arbitrary constant C_3 is found to be given by (3.73).

Now, we need to check that the stress solution in the inner rigid region satisfies the strict inequality of the yield condition, namely (3.60). Thus, from (3.62), (3.74) and (3.75), we find that (3.60) becomes

$$\left(\frac{z}{r}\right)^3 + b^3 > 0, \tag{3.78}$$

where we have used the relation $b^3D(1-2E) = -E^2$. In this case, (3.78) gives rise to only one possible condition $z/r < -b$, which is satisfied throughout the entire inner rigid region, due to the inner rigid region being bounded by the surface $z/r = -b$, as shown in Figure 3. We note that we only consider $r \geq 0$ because we have assumed a symmetrical stock pile.

Finally, we note that for both the linear and singular form of the stresses, (3.61) and (3.62) respectively, the expressions for the determination of s_2 are essentially identical, namely (3.70) and (3.77). This is not surprising, because if we consider (3.61) and (3.62) along $z = -br$, then they are identical provided

$$A = -b^2D, \quad B = -2bD, \quad C = E, \tag{3.79}$$

and as such, the form of the mathematical solution in the inner rigid region will be similar.

4 Results and conclusions

We have utilized recently determined exact parametric solutions, derived in Hill and Cox [18] and Cox and Hill [20], to determine the stress distributions within two-dimensional wedge and three-dimensional conical shaped stock piles, assuming a highly frictional granular solid. This work extends that of Hill and Cox [14] to the case of $\phi = \pi/2$. Following [14], we assume the stock piles are composed of two regions; an outer yield region, where the material is assumed to be at the limiting equilibrium point of yield, and an inner rigid region, where the material is assumed to be in equilibrium, but not at yield. Hill and Cox [14] determine solutions in two and three-dimensions for $\sin\phi \neq 1$, while here we specifically assume that $\sin\phi = 1$. In this event, the solution procedure outlined in [14] does not apply.

For the two-dimensional plane strain wedge shaped stock pile, we assume that the top surface of the outer yield region is given by $y = -a|x|$, and the boundary between the outer yield region and the inner rigid region is given by $y = -b|x|$, where $0 < a < b < \infty$ and a is assumed to be known. In the outer yield region, we utilize the previously determined exact parametric solution (3.29), and assume that the top surface is stress free. In the inner rigid region, there is no unique procedure for determining the form of the stress distribution. Here, we assume the stresses are extended into the inner rigid region in such a manner that along the boundary between the two regions, the stresses are continuous and of the same form. This procedure gives rise to solutions that are not uniquely determined, but depend on an arbitrary parameter B, which lies in the range $1/2 < B < 1$. Thus, for B prescribed and for given external dimensions of the stock pile of height h and half length h/a, we are able to analytically determine the stress distribution throughout the entire stock pile. Figure 4 shows a two-dimensional stock pile comprising of a rigid inner region and an outer yield region, with the variation of $\sigma_{xx}/\rho g h$, $\sigma_{xy}/\rho g h$ and $\sigma_{yy}/\rho g h$ along the base $y = -h$ with respect to x/h, where $a = 1$ and $B = 0.75$. We note that the horizontal and vertical forces along the base of the stock pile, namely along $y = -h$, are given by $\sigma_x = \sigma_{xy}$ and $\sigma_y = \sigma_{yy}$ respectively.

For the three-dimensional axially symmetric conical shaped stock pile, we assume that the top surface of the outer yield region is given by $z = -a|r|$, and the boundary between the outer yield region and the inner rigid region is given by $z = -b|r|$, where $0 < a < b < \infty$ and a is assumed to be known. In the outer yield region, we utilize the previously determined exact parametric solution (3.48), and assume that the top surface is stress free. In the inner rigid region, we again note there is no unique procedure for determining the form of the stress distribution. Our procedure gives rise to solutions that are not uniquely determined, and in the three-dimensional case, we are able to determine at least two possible forms of the stress distribution, which depend on an arbitrary parameter C (or E), which lies in the range $1/3 <$

C(or E) $< 1/2$. Thus, once C (or E) is prescribed and for given external dimensions of the stock pile h and half length h/a, we are able to analytically determine the stress solution throughout the entire stock pile. Figure 5 shows a three-dimensional stock pile comprising of a rigid inner region and an outer yield region, with the variation of $\sigma_{rr}/\rho gh$, $\sigma_{rz}/\rho gh, \sigma_{zz}/\rho gh$ and $\sigma_{\theta\theta}/\rho gh$ along the base $z = -h$ with respect to r/h, according to the linear form of the stresses given by (3.61), where $a = 1$ and $C = 0.49$. Similarly, Figure 6 shows a three-dimensional stock pile comprising of a rigid inner region and an outer yield region, with the variation of $\sigma_{rr}/\rho gh$, $\sigma_{rz}/\rho gh$ and $\sigma_{zz}/\rho gh$ along the base $z = -h$ with respect to r/h, according to the singular form of the stresses given by (3.62), where $a = 1, E = 0.49$ and $\sigma_{\theta\theta}$ is zero throughout the entire stock pile. We note that the horizontal and vertical forces along the base of the stock pile, namely along $z = -h$, are given by $\sigma_r = \sigma_{rz}$ and $\sigma_z = \sigma_{zz}$ respectively.

From the figures, it is clear that the model predicts the maximum in magnitude of the horizontal and vertical stresses are located at the extremities of the stock pile. This result, at first counterintuitive, perhaps occurs due to the high value of the angle of internal friction, where an internal arch has formed within the stock pile and is entirely consistent with the results given in [14]. For comparison, we utilize the scheme given in Hill and Cox [14] for values of β approaching unity and compare them with our prediction for $\beta = 1$. In this case, we find $a = 0.64$ and again we assume $B = 0.75$, where Figure 7 shows the two-dimensional horizontal and vertical force distributions for $\beta = \cos\alpha, 0.75, 0.9$ and 0.95, where $\alpha = 287\pi/900$ is the half angle of the stock pile. It is clear that as β tends to unity, the maximum in magnitude of the stresses approaches the extremities of the stock pile, agreeing with the results presented here. In fact, from Hill and Cox [14] we see that the maximum is located at the boundary between the two regions, given by $\theta = \gamma$, where γ must satisfy the condition $\gamma > (\psi_0 + \pi - \arccos(\beta \cos\psi_0))/2$, where ψ_0 is the constant of integration for the Sokolovsky solution. Thus, if $\beta = 1$ then $\gamma > \pi/2$, or in other words, the boundary between the two regions has moved to the extremities, and according to [14], the outer yield region vanishes. We also note the assumed form of the stresses in the inner dead region according to Hill and Cox [14] is such that they are linear in both x and y, whereas here we have assumed the simplest form of the stresses such that the stresses in the outer yield region may be extended continuously into the inner rigid region. This gives rise to the constant value of the vertical stress in the inner rigid region. We observe that as a direct consequence of satisfying the zero traction condition on the upper surface of the outer yield region, the generally positive quantity q, given by (3.29) and (3.48) for the two geometries respectively, is necessarily infinite on the stress-free surface. Finally, from the singular form of the stresses in the inner rigid region of the three-dimensional axially symmetric stock pile, although there is a singularity occurring in $\sigma_{rr} = \rho g D z^2/r$, we emphasize that this is a weak singularity in the sense that the integrated stress vectors are finite, and that this singularity plays no role in the determination of the forces along the base.

Acknowledgements
The support from the Australian Research Council through the Large Grant Scheme, a Senior Research Fellowship for JMH and a Australian Post-Doctoral Fellowship for GMC is gratefully acknowledged. The first author gratefully acknowledges the Institute for the Promotion of teaching Science and Technology (IPST), Thailand, for financial support.

References

[1] J. Smid and J. Novosad, Pressure distribution under heaped bulk solids, *I. Chem. E. Symp.* **63** (1981), D3/V/1-12.

[2] A. Watson, The perplexing puzzle posed by a pile of apples, *New Scientist* **1799** (1991), 19.

[3] A. Watson, Searching for the sand-pile pressure dip, *Science* **273** (1996), 579-580.

[4] D. F. Bagster, An idealised model of granular material behaviour in ore heaps and hoppers, *J. Powder Bulk Solids Tech.* **2** (1978), 42-46.

[5] D. F. Bagster, A randomised model of granular material in an ore heap, *J. Powder Bulk Solids Tech.* **6** (1982), 1-3.

[6] D. F. Bagster, The development of a microscopic model of granular material behaviour in a heap, *3rd Int. Conf. Bulk Materials, Storage, Handling and Transportation*, 27-29 June 1989, Newcastle, Australia. (1989), 24-32.

[7] D. F. Bagster and E. Li, The stresses beneath a heap of blocks. *Int. Conf. Bulk Materials Handling and Transportation*, 6-8 July 1992, Wollongong, Australia. (1992), 419-423.

[8] D. F. Bagster and E. Li, A three-dimensional random heap model of hard spheres. *Nat. Conf. Bulk Materials Handling*, 22-25 Sept. 1993, Yeppoon, Australia. (1993), 67-71.

[9] D. F. Bagster and R. Kirk, Computer generation of a model to simulate granular material behaviour. *J. Powder Bulk Solids Tech.* **9** (1985), 19-24.

[10] G. R. Brooks and D. F. Bagster, The slack contact model of stresses in a heap of particles. *J. Powder Bulk Solids Tech.* **8** (1984), 18-28.

[11] K. Liffman, D. Y. C. Chan and B. D. Hughes, Force distribution in a two dimensional sand-pile. *J. Powder Tech.* **72** (1992), 255-267.

[12] L. Vanel, D. Howell, D. Clark, R. P. Behringer and E. Clement, Memories in sand: Experimental tests of construction history on stress distributions under sandpiles, *Phys. Rev. E.* **60** (1999), R5040-R5043.

[13] S. B. Savage, Problems in the statics and dynamics of granular materials. *Proc. "Powders and Grains '97"*, (eds. Behringer, R. P. & Jenkins, J. T. Balkema, Rotterdam) (1997), 185-194.

[14] J. M. Hill and G. M. Cox, The force distribution at the base of sand-piles. *Developments in Theoretical Geomechanics*, The John Booker Memorial Symposium, (eds. Smith, D. W. & Carter, J. P.), (2000) 43-61.

[15] F. Cantelaube and J. D. Goddard, Elastoplastic arching in 2D heaps, *Powders and Grains*, Proc. Third Int. Conf., Durham, NC, USA, 18-23 May, 1997, (ed. Behringer, R. P. & Jenkins, J. T. Balkema, Rotterdam) (1997), 231-234

[16] F. Cantelaube, A. K. Didwania and J. D. Goddard, Elasto-plastic arching in two dimensional granular heaps, *Physics of Dry Granular Media*. Proc. NATO ASI, Cargese, France, 15-26 1997, (ed. Herrmann, H. J., Hovi, J. P. & Luding, S. Kluwer, Dordrecht) (1997), 123-127.

[17] A. D. Didwania, F. Cantelaube and J. D. Goddard, Static multiplicity of stress states in granular heaps. *Proc. R. Soc. Lond. A.* **456** (2000), 2569 - 2588.

[18] J. M. Hill and G. M. Cox, On the problem of the determination of force distributions in granular heaps using continuum theory. *Q. J. Mech. Appl. Math.* **55** (2002), 655-668.

[19] V. V. Sokolovsky, Statics of granular materials. Pergamon, Oxford (1965).

[20] G. M. Cox and J. M. Hill, Some exact mathematical solutions for granular stock piles and granular flow in hoppers. *Mathematics and Mechanics of Solids.* **8** (2003), 21-50.

[21] J. M. Hill and G. M. Cox, An exact parametric solution for granular flow in a converging wedge. *Zeitschrift fur angewandte Mathematik und Physik (ZAMP).* **52** (2001), 657-668.

[22] N. Thamwattana and J. M. Hill, Force distributions at the bases of curved highly frictional granular stockpile. *Q. J. Maths. Appl. Mechs.* (2003), submitted for publication.

[23] N. Thamwattana and J. M. Hill, Analytical solutions for tapering quadratic and cubic rat-holes in highly frictional granular solids. *Int. J. of Solids and Structures.* **40** (2003), 5923-5948.

[24] Australian Standard, Loads on bulk solids containers, Standards Association of Australia. ISBN 0733707335, **AS 3774** (1996), 23.

[25] S. W. Perkins, Non-linear limit analysis for the bearing capacity of highly frictional soils. *2nd Congress on Computing in Civil Engineering*, ASCE, Atlanta, 4 June 1995, **1** (1994), 629-636.

[26] S. W. Perkins, Bearing capacity of highly frictional material. *ASTM Geotechnical Testing Journal.* **18** (1995), 450-462.

[27] S. Sture, Constitutive issues in soil liquefaction, *Physics and Mechanics of Soil Liquefaction*, (eds. Lade, P. V. & Yanamuro, J. A.; Balkema, Rotterdam, 1999) (1999), 133-143.

[28] K. M. Lynch and M. T. Mason, Pulling by pushing, slip with infinite friction, and perfectly rough surfaces. *Int. Conf. on Robotics and Automation*, IEEE, May 2-6, Atlanta, 1993, **1** (1993), 745-751.

[29] K. M. Lynch and M. T. Mason, Pulling by pushing, slip with infinite friction, and perfectly rough surfaces. *Int. J. of Robotics Research*, **14** (1995), 174-183.

[30] A. J. M. Spencer, Deformation of ideal granular materials, *Mechanics of Solids: The Rodney Hill 60th Anniversary Volume* (eds. Hopkins, H. G. & Sewell, M. J.; Oxford, Pergamon), (1982), 607-652.

[31] G. M. Cox, J. M. Hill and N. Thamwattana, An analytical solution for a sloping rat-hole in a highly frictional solid. *Phil. Trans. Roy. Soc. Lond.* (2003), submitted for publication.

Figure captions

Figure 1. Mohr circle diagrams for general values of the angle of internal friction and for the special case of ninety degrees. (a) $\phi < \pi/2$ and (b) $\phi = \pi/2$.

Figure 2. (a) Schematic of the two-dimensional plane strain wedge shaped stock pile, comprising of an inner rigid region and a outer yield region. (b) Coordinate system for the two-dimensional stock pile.

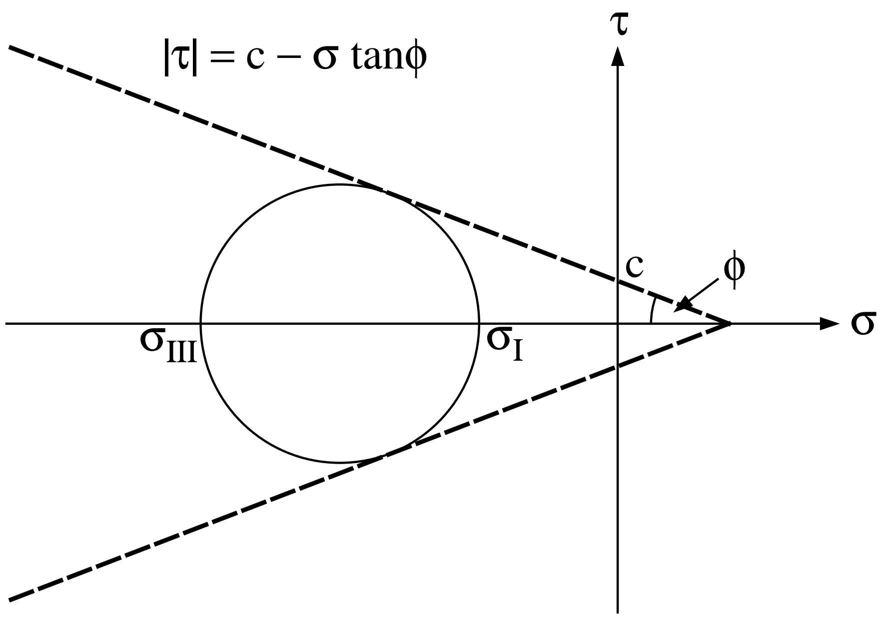
Figure 3. (a) Schematic of the three-dimensional axially symmetric cone shaped stock pile, comprising of an inner rigid region and a outer yield region. (b) Coordinate system for the three-dimensional stock pile.

Figure 4. (a) The outer yield region and inner rigid region surfaces for the two-dimensional stock pile. Variation of the stresses along the base $y = -h$: (b) $\sigma_{xx}/\rho gh$, (c) $\sigma_{xy}/\rho gh$ and (d) $\sigma_{yy}/\rho gh$ with x/h, where $B = 0.75$ and $a = 1$.

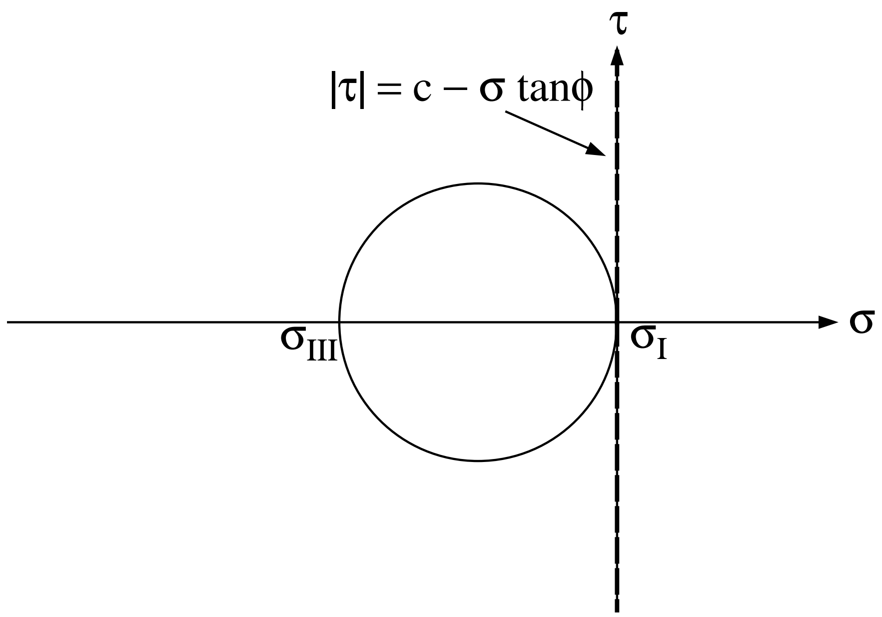
Figure 5. (a) The outer yield region and inner rigid region surfaces for the three-dimensional stock pile. Variation of the stresses along the base $z = -h$ according to (3.61): (b) $\sigma_{rr}/\rho gh$, (c) $\sigma_{rz}/\rho gh$, (d) $\sigma_{zz}/\rho gh$ and (e) $\sigma_{\theta\theta}/\rho gh$ with r/h, where $C = 0.49$ and $a = 1$.

Figure 6. (a) The outer yield region and inner rigid region surfaces for the three-dimensional stock pile. Variation of the stresses along the base $z = -h$ according to (3.62): (b) $\sigma_{rr}/\rho gh$, (c) $\sigma_{rz}/\rho gh$ and (d) $\sigma_{zz}/\rho gh$ with r/h, where $E = 0.49$ and $a = 1$.

Figure 7. Comparison of the predicted horizontal and vertical force distributions from our results with those from Hill and Cox [14] as β tends to unity. ((a). horizontal and (b). vertical).

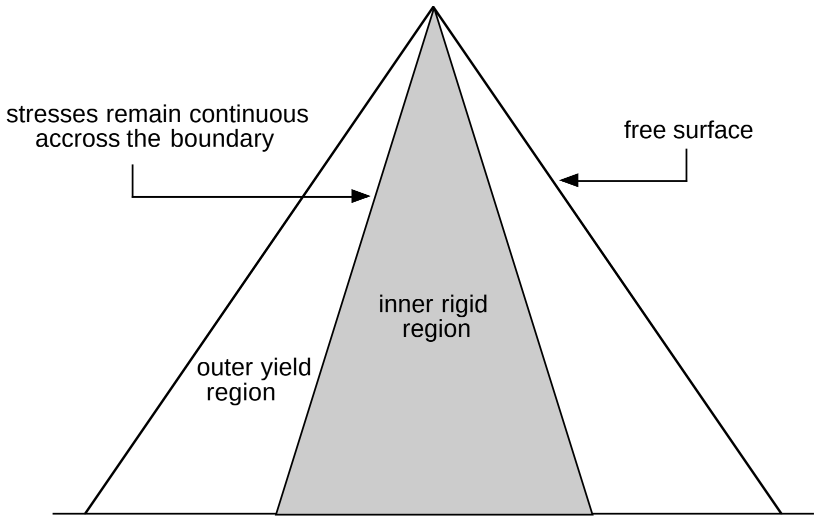


(a)

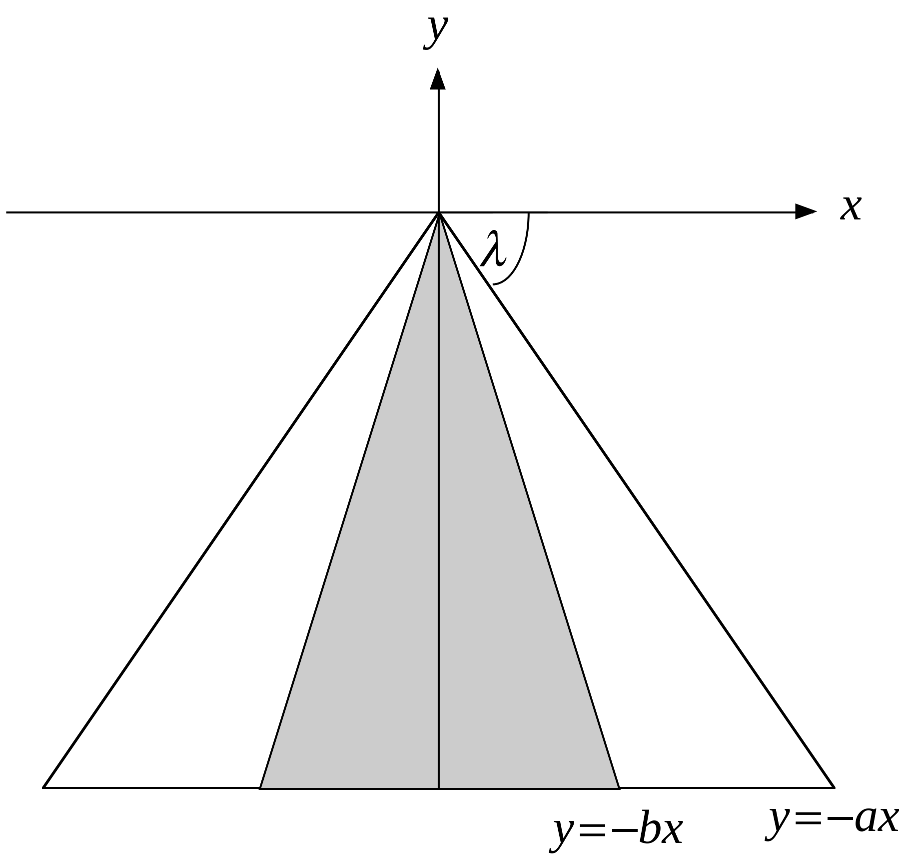


(b)

Figure 1

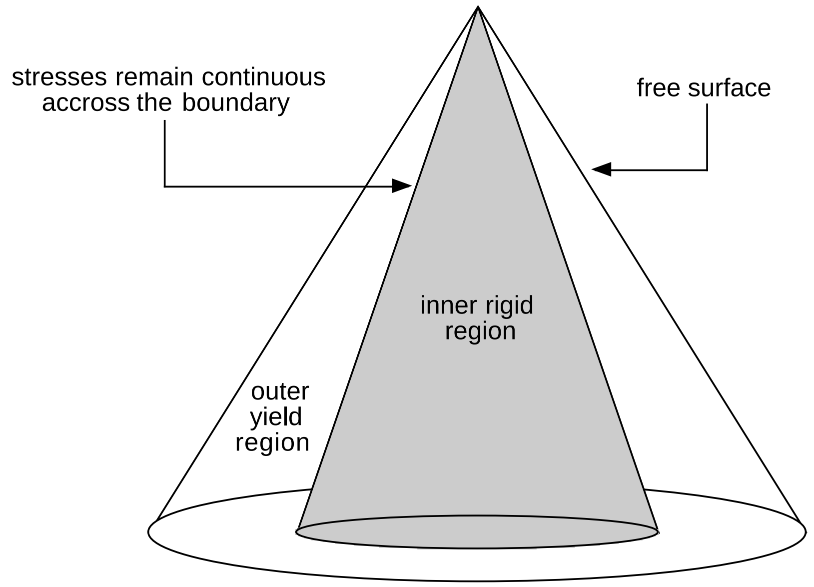

stresses remain continuous
accross the boundary
free surface
inner rigid
region
outer yield
region

(a)


y
x
λ
y=−bx
y=−ax

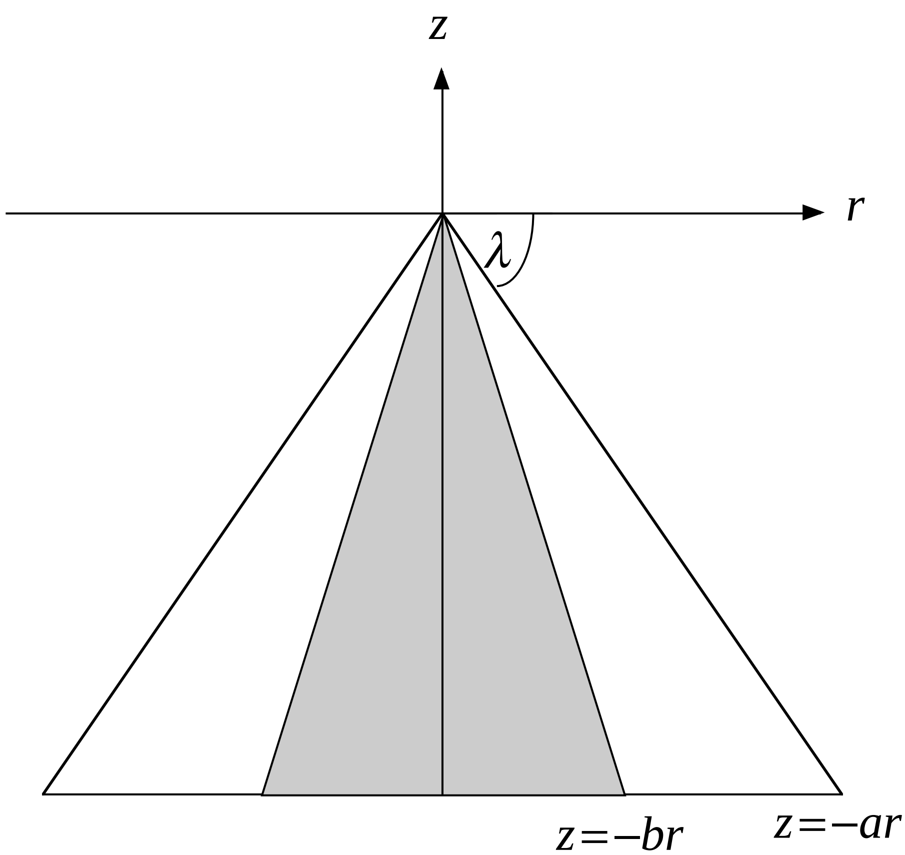
(b)

Figure 2



stresses remain continuous
accross the boundary
free surface
inner rigid
region
outer
yield
region

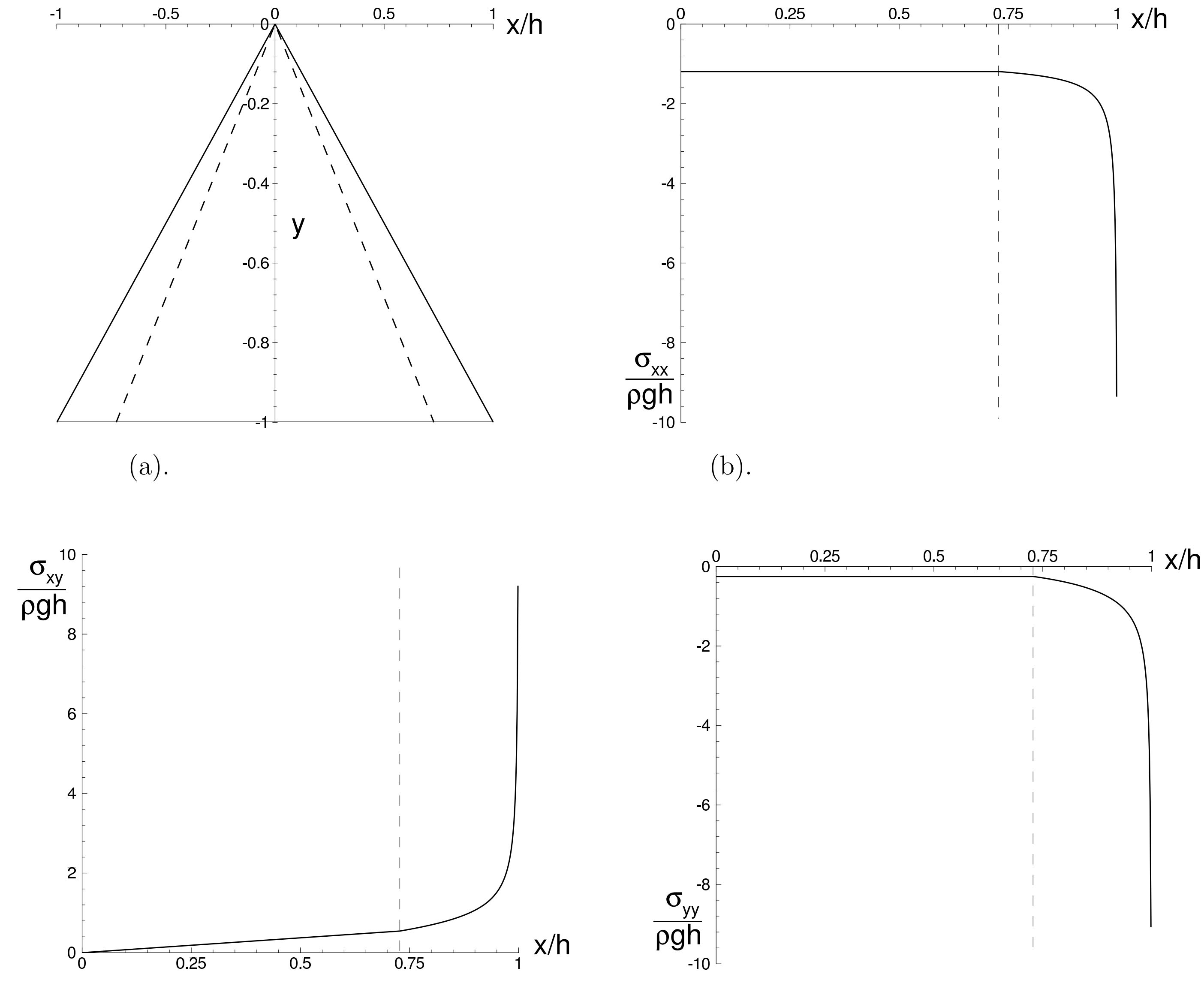
(a)



z
r
λ
z=−br
z=−ar

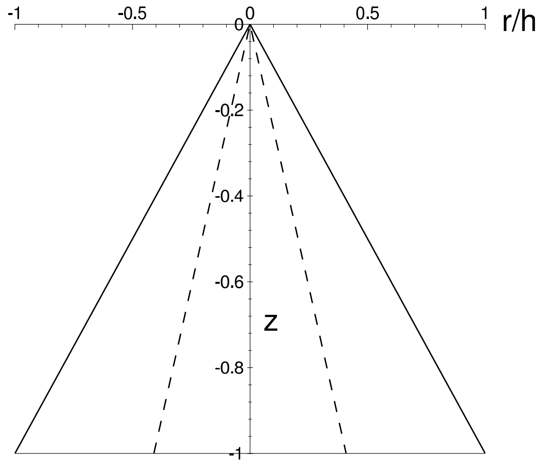
(b)

Figure 3

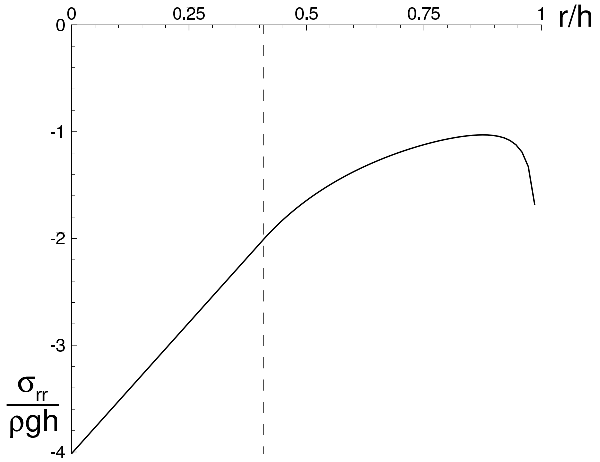


x/h
y
$\frac{\sigma_{xx}}{\rho g h}$
$\frac{\sigma_{xy}}{\rho g h}$
$\frac{\sigma_{yy}}{\rho g h}$
(a).
(b).
(c).
(d).

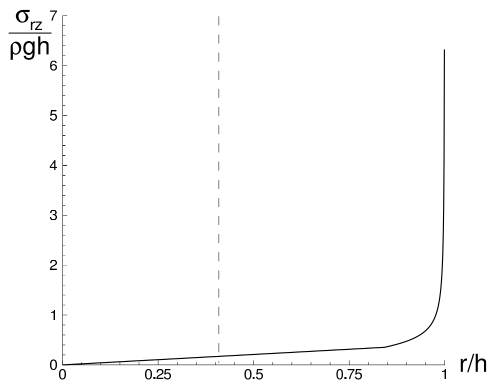
Figure 4



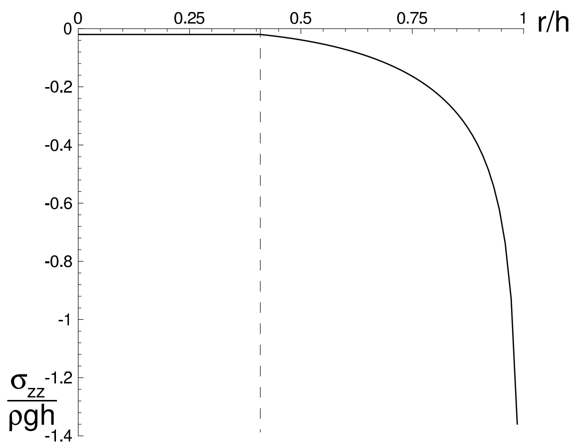
(a)



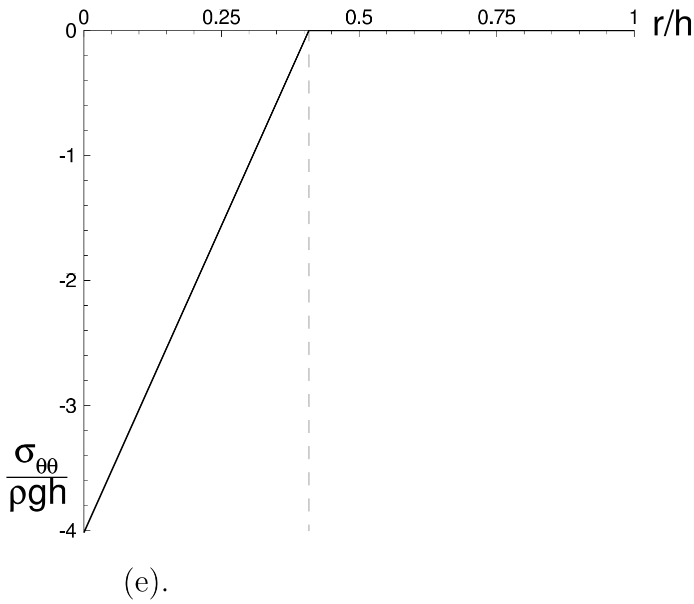
(b).



(c).



(d).



(e).

Figure 5

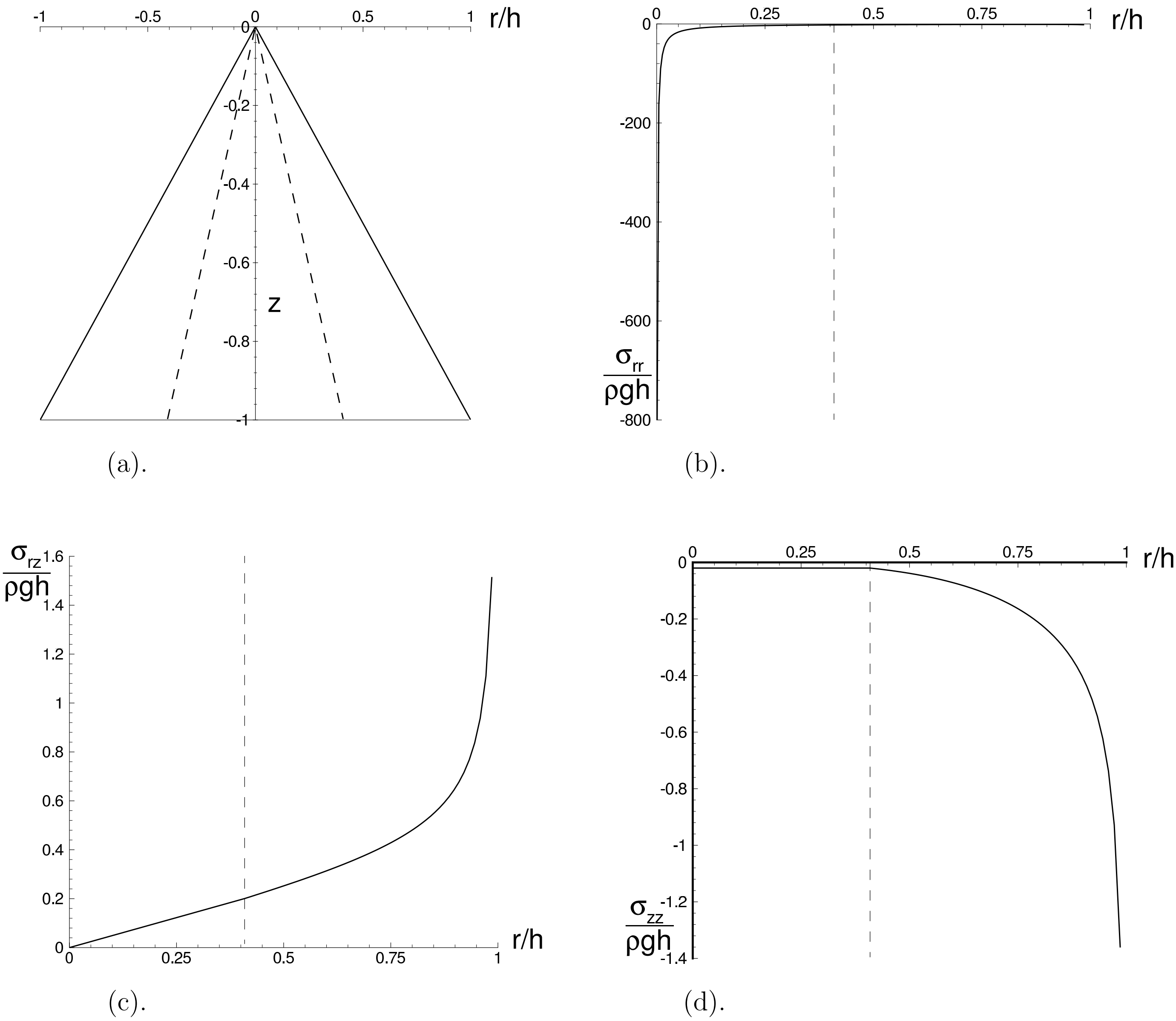
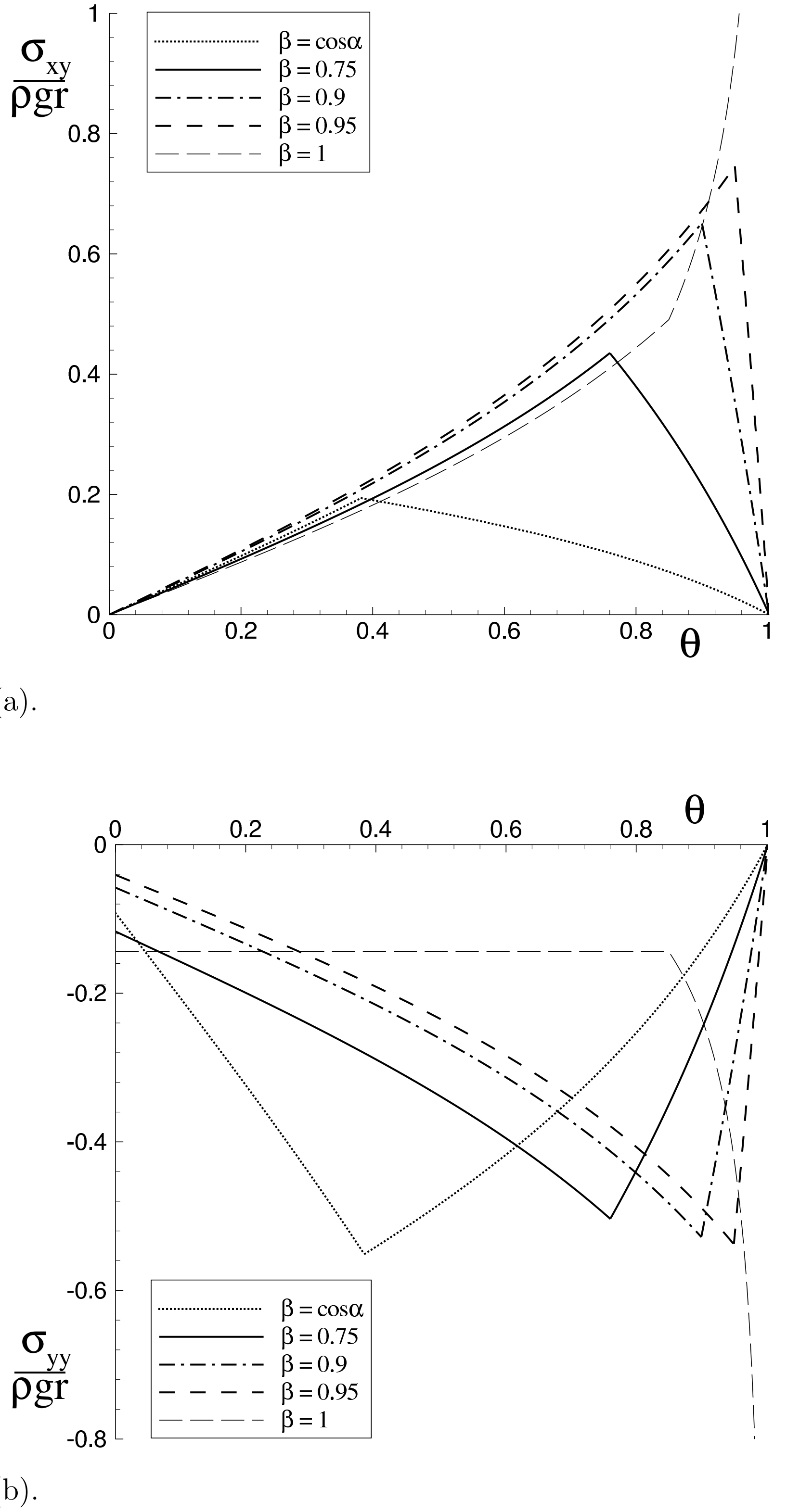

r/h
z
(a).
r/h
$\frac{\sigma_{rr}}{\rho g h}$
(b).
$\frac{\sigma_{rz}}{\rho g h}$
r/h
(c).
r/h
$\frac{\sigma_{zz}}{\rho g h}$
(d).

Figure 6



σxy / ρgr
θ
β = cosα
β = 0.75
β = 0.9
β = 0.95
β = 1
0
0.2
0.4
0.6
0.8
1

(a).

σyy / ρgr
θ
β = cosα
β = 0.75
β = 0.9
β = 0.95
β = 1
0
-0.2
-0.4
-0.6
-0.8
0.2
0.4
0.6
0.8
1

(b).

Figure 7